PERGAMON

Automatica 38 (2002) 3–20

automatica

www.elsevier.com/locate/automatica

# The explicit linear quadratic regulator for constrained systems☆

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Received 24 September 1999; revised 9 October 2000; received in final form 16 June 2001

*We present a technique to compute the explicit state-feedback solution to both the finite and infinite horizon linear quadratic optimal control problem subject to state and input constraints. We show that this closed form solution is piecewise linear and continuous. As a practical consequence of the result, constrained linear quadratic regulation becomes attractive also for systems with high sampling rates, as on-line quadratic programming solvers are no more required for the implementation.*

**Abstract**

For discrete-time linear time invariant systems with constraints on inputs and states, we develop an algorithm to determine explicitly, the state feedback control law which minimizes a quadratic performance criterion. We show that the control law is piece-wise linear and continuous for both the finite horizon problem (model predictive control) and the usual infinite time measure (constrained linear quadratic regulation). Thus, the on-line control computation reduces to the simple evaluation of an explicitly defined piecewise linear function. By computing the inherent underlying controller structure, we also solve the equivalent of the Hamilton–Jacobi–Bellman equation for discrete-time linear constrained systems. Control based on on-line optimization has long been recognized as a superior alternative for constrained systems. The technique proposed in this paper is attractive for a wide range of practical problems where the computational complexity of on-line optimization is prohibitive. It also provides an insight into the structure underlying optimization-based controllers. 

*Keywords:* Piecewise linear controllers; Linear quadratic regulators; Constraints; Predictive control

## 1. Introduction

As we extend the class of system descriptions beyond the class of linear systems, *linear systems with constraints* are probably the most important class in practice and the most studied. It is well accepted that for these systems, in general, stability and good performance can only be achieved with a non-linear control law. The most popular approaches for designing non-linear controllers for linear systems with constraints fall into two categories: anti-windup and model predictive control.

Anti-windup schemes assume that a well functioning linear controller is available for small excursions from the nominal operating point. This controller is augmented by the anti-windup scheme in a somewhat ad hoc fashion, to take care of situations when constraints are met. Kothare et al. (1994) reviewed numerous apparently different anti-windup schemes and showed that they differ only in their choice of two static matrix parameters. The least conservative stability test for these schemes can be formulated in terms of a linear matrix inequality (LMI) (Kothare & Morari, 1999). The systematic and automatic *synthesis* of anti-windup schemes which guarantee closed-loop stability and achieve some kind of optimal

☆This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Per-Olof Gutman under the direction of Editor Tamer Basar.

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0005-1098/02/$ - see front matter 
PII: S0005-1098(01)00174-1

performance, has remained largely elusive, though some promising steps were achieved recently (Mulder, Kothare, & Morari, 1999; Teel & Kapoor, 1997). Despite these drawbacks, anti-windup schemes are widely used in practice because in most SISO situations, they are simple to design and work adequately.

Model predictive control (MPC) has become the accepted standard for complex constrained multivariable control problems in the process industries. Here, at each sampling time, starting at the current state, an open-loop optimal control problem is solved over a finite horizon. At the next time step, the computation is repeated starting from the new state and over a shifted horizon, leading to a moving horizon policy. The solution relies on a linear dynamic model, respects all input and output constraints, and optimizes a quadratic performance index. Thus, as much as a quadratic performance index together with various constraints can be used to express true performance objectives, the performance of MPC is excellent. Over the last decade, a solid theoretical foundation for MPC has emerged so that in real life, large-scale MIMO applications controllers with non-conservative stability guarantees can be designed routinely and with ease (Qin & Badgwell, 1997). The big drawback of the MPC is the relatively formidable on-line computational effort, which limits its applicability to relatively slow and/or small problems.

In this paper, we show how to move all the computations necessary for the implementation of MPC off-line, while preserving all its other characteristics. This should largely increase the range of applicability of MPC to problems where anti-windup schemes and other ad hoc techniques dominated up to now. Moreover, such an explicit form of the controller provides additional insight for better understanding of the control policy of MPC.

We also show how to solve the equivalent of the Hamilton–Jacobi–Bellman equation for discrete-time linear constrained systems. Rather than gridding the state space in some ad hoc fashion, we discover the inherent underlying controller structure and provide its most efficient parameterization.

The paper is organized as follows. The basics of MPC are reviewed first to derive the quadratic program which needs to be solved to determine the optimal control action. We proceed to show that the form of this quadratic program is maintained for various practical extensions of the basic setup, for example, trajectory following, suppression of disturbances, time-varying constraints and also the output feedback problem. As the coefficients of the linear term in the cost function and the right-hand side of the constraints depend linearly on the current state, the quadratic program can be viewed as a *multi-parametric quadratic program* (mp-QP). We analyze the properties of mp-QP, develop an efficient algorithm to solve it, and show that the optimal solution is a piecewise affine function of the state (confirming other investigations on the form of MPC laws (Chiou & Zafiriou, 1994; Tan, 1991; Zafiriou, 1990; Johansen, Petersen, & Slupphaug, 2000).

The problem of synthesizing stabilizing feedback controllers for linear discrete-time systems, subject to input and state constraints, was also addressed in Gutman (1986) and Gutman and Cwikel (1986). The authors obtain a piecewise linear feedback law defined over a partition of the set of states $X$ into simplicial cones, by computing a feasible input sequence for each vertex via linear programming (this technique was later extended in Blanchini, 1994). Our approach provides a piecewise affine control law which not only ensures feasibility and stability, but is also optimal with respect to LQR performance.

The paper concludes with a series of examples which illustrate the different features of the method. Other material related to the contents of this paper can be found at `http://control.ethz.ch/~bemporad/explicit`.

## 2. Model predictive control

Consider the problem of regulating to the origin the discrete-time linear time invariant system

$$\begin{cases} x(t+1) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

while fulfilling the constraints

$$y_{\min} \leqslant y(t) \leqslant y_{\max}, \quad u_{\min} \leqslant u(t) \leqslant u_{\max} \tag{2}$$

at all time instants $t \geqslant 0$. In (1)–(2), $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ are the state, input, and output vectors, respectively, $y_{\min} \leqslant y_{\max}$ ($u_{\min} \leqslant u_{\max}$) are $p(m)$-dimensional vectors[1], and the pair $(A, B)$ is stabilizable.

Model predictive control (MPC) solves such a constrained regulation problem in the following way. Assume that a full measurement of the state $x(t)$ is available at the current time $t$. Then, the optimization problem

$$\min_{U \triangleq \{u_t, \ldots, u_{t+N_u-1}\}} \left\{ J(U, x(t)) = x'_{t+N_y|t} P x_{t+N_y|t} + \sum_{k=0}^{N_y-1} [x'_{t+k|t} Q x_{t+k|t} + u'_{t+k} R u_{t+k}] \right\},$$

[1] The results of this paper hold also for the more general mixed constraints $D_1 x(t) + D_2 u(t) \leqslant d$.

$$
\begin{aligned}
\text{s.t.}\quad & y_{\min} \leqslant y_{t+k|t} \leqslant y_{\max}, \quad k = 1, \ldots, N_c, \\
& u_{\min} \leqslant u_{t+k} \leqslant u_{\max}, \quad k = 0, 1, \ldots, N_c, \\
& x_{t|t} = x(t), \\
& x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k}, \quad k \geqslant 0, \\
& y_{t+k|t} = Cx_{t+k|t}, \quad k \geqslant 0, \\
& u_{t+k} = Kx_{t+k|t}, \quad N_u \leqslant k < N_y,
\end{aligned} \tag{3}
$$

is solved at each time $t$, where $x_{t+k|t}$ denotes the predicted state vector at time $t+k$, obtained by applying the input sequence $u_t, \ldots, u_{t+k-1}$ to model (1) starting from the state $x(t)$. In (3), we assume that $Q = Q' \succcurlyeq 0$, $R = R' \succ 0$, $P \succcurlyeq 0$, $(Q^{1/2}, A)$ detectable (for instance, $Q = C'C$ with $(C, A)$ detectable), $K$ is some feedback gain, $N_y$, $N_u$, $N_c$ are the output, input, and constraint horizons, respectively, with $N_u \leqslant N_y$ and $N_c \leqslant N_y - 1$ (the results of this paper also apply for $N_c \geqslant N_y$).

One possibility is to choose $K = 0$ (Rawlings & Muske, 1993), and $P$ as the solution of the Lyapunov equation

$$P = A'PA + Q. \tag{4}$$

The choice is only meaningful for open-loop stable systems, as it implies that after $N_u$ time steps, the control is turned off.

Alternatively, one can set $K = K_{LQ}$ (Chmielewski & Manousiouthakis, 1996; Scokaert & Rawlings, 1998), where $K_{LQ}$ and $P$ are, the solutions of the unconstrained infinite horizon LQR problem with weights $Q, R$

$$K_{LQ} = -(R + B'PB)^{-1}B'PA, \tag{5a}$$

$$P = (A + BK_{LQ})'P(A + BK_{LQ}) + K'_{LQ}RK_{LQ} + Q. \tag{5b}$$

This choice of $K$ implies that, after $N_u$ time steps, the control is switched to the unconstrained LQR. With $P$ obtained from (5), $J(U, x(t))$ measures the settling cost of the system from the present time $t$ to infinity under this control assumption. In addition, if one guarantees that the predicted input and output evolutions do not violate the constraints also at later time steps $t+k$, $k = N_u + 1, \ldots, +\infty$, this strategy is indeed the optimal infinite horizon constrained LQ policy. This point will be addressed in Section 3.

The MPC control law is based on the following idea: At time $t$, compute the optimal solution $U^*(t) = \{u_t^*, \ldots, u_{t+N_u-1}^*\}$ to problem (3), apply

$$u(t) = u_t^* \tag{6}$$

as input to system (1), and repeat the optimization (3) at time $t+1$, based on the new state $x(t+1)$. Such a control strategy is also referred to as *moving* or *receding horizon*.

The two main issues regarding this policy are feasibility of the optimization problem (3) and stability of the resulting closed-loop system. When $N_c < \infty$, there is no guarantee that the optimization problem (3) will remain feasible at all future time steps $t$, as the system might enter "blind alleys" where no solution to problem (3) exists. On the other hand, setting $N_c = \infty$ leads to an optimization problem with an infinite number of constraints, which is impossible to handle. If the set of feasible state + input vectors is bounded and contains the origin in its interior, by using arguments from maximal output admissible set theory (Gilbert & Tin Tan, 1991), for the case $K = 0$ Bemporad (1998) showed that a finite constraint horizon, $N_c$, can be chosen without loss of guarantees of constraint fulfillment, and a similar argument can be repeated for the case $K = K_{LQ}$. Similar results about the choice of the smallest $N_c$ ensuring feasibility of the MPC problem (3) at all time instants were also proved in Gutman and Cwikel (1986) and have been extended recently in Kerrigan and Maciejowski (2000).

The stability of MPC feedback loops was investigated by numerous researchers. Stability is, in general, a complex function of the various tuning parameters $N_u$, $N_y$, $N_c$, $P$, $Q$, and $R$. For applications, it is most useful to impose some conditions on $N_y$, $N_c$ and $P$, so that stability is guaranteed for all $Q \succcurlyeq 0$, $R \succ 0$. Then, $Q$ and $R$ can be freely chosen as tuning parameters to affect performance. Sometimes, the optimization problem (3) is augmented with a so-called "stability constraint" (see Bemporad & Morari (1999) for a survey of different constraints proposed in the literature). This additional constraint imposed over the prediction horizon, explicitly forces the state vector either to shrink in some norm or to reach an invariant set at the end of the prediction horizon.

Most approaches for proving stability follow in spirit the arguments of Keerthi and Gilbert (1988) who establish the fact that under some conditions, the value function $V(t) = J(U^*(t), t)$ attained at the minimizer $U^*(t)$ is a Lyapunov function for the system. Below, we recall a simple stability result based on such a Lyapunov argument (see also Bemporad, Chisci, & Mosca, 1994).

**Theorem 1.** *Let* $N_y = \infty$, $K = 0$ *or* $K = K_{LQ}$, *and* $N_c < \infty$ *be sufficiently large for guaranteeing existence of feasible input sequences at each time step.*[2] *Then, the MPC law* (3)–(6) *asymptotically stabilizes system* (1) *while*

[2] As discussed above, several techniques exist to compute constraint horizons $N_c$ which guarantee feasibility, see Bemporad (1998), Gilbert and Tin Tan (1991), Gutman and Cwikel (1986) and Kerrigan and Maciejowski (2000).

*enforcing the fulfillment of the constraints* (2) *from all initial states* $x(0)$ *such that* (3) *is feasible at* $t = 0$.

Theorem 1 ensures stability, provided that the optimization problem (3) is feasible at time $t = 0$. The problem of determining the set of initial conditions $x(0)$ for which (3) is feasible has been addressed in Gutman and Cwikel (1986), and more recently in Kerrigan and Maciejowski (2000).

### 2.1. MPC computation

By substituting $x_{t+k|t} = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{t+k-1-j}$, Eq. (3) can be rewritten as

$$V(x(t)) = \frac{1}{2} x'(t) Y x(t) + \min_U \left\{ \frac{1}{2} U' H U + x'(t) F U, \right.$$
$$\left. \text{s.t.} \quad GU \leqslant W + Ex(t) \right\}, \tag{7}$$

where the column vector $U \triangleq [u'_t, \dots, u'_{t+N_u-1}]' \in \mathbb{R}^s$, $s \triangleq mN_u$, is the optimization vector, $H = H' \succ 0$, and $H, F, Y, G, W, E$ are easily obtained from $Q$, $R$, and (3) (as only the optimizer $U$ is needed, the term involving $Y$ is usually removed from (7)).

The optimization problem (7) is a quadratic program (QP). Since the problem depends on the current state $x(t)$, the implementation of MPC requires the on-line solution of a QP at each time step. Although efficient QP solvers based on active-set methods and interior point methods are available, computing the input $u(t)$ demands significant on-line computation effort. For this reason, the application of MPC has been limited to "slow" and/or "small" processes.

In this paper, we propose a new approach to implement MPC, where the computation effort is moved off-line. The MPC formulation described in Section 2 provides the control action $u(t)$ as a function of $x(t)$ *implicitly* defined by (7). By treating $x(t)$ as a vector of parameters, our goal is to solve (7) off-line, with respect to all the values of $x(t)$ of interest, and make this dependence *explicit*.

The operations research community has addressed parameter variations in mathematical programs at two levels: *sensitivity analysis*, which characterizes the change of the solution with respect to small perturbations of the parameters, and *parametric programming*, where the characterization of the solution for a full range of parameter values is sought. In the jargon of operations research, programs which depend only on one scalar parameter are referred to as *parametric programs*, while problems depending on a vector of parameters as *multi-parametric programs*. According to this terminology, (7) is a multi-parametric quadratic program (mp-QP). Most of the literature deals with parametric problems, but some authors have addressed the multi-parametric case (Acevedo & Pistikopoulos, 1999; Dua & Pistikopoulos, 1999, 2000; Fiacco, 1983; Gal, 1995). In Section 4, we will describe an algorithm to solve mp-QP problems. To the authors' knowledge, no algorithm for solving mp-QP problems has appeared in the literature. Once the multi-parametric problem (7) has been solved off line, i.e. the solution $U_t^* = U^*(x(t))$ of (7) has been found, the model predictive controller (3) is available explicitly, as the optimal input $u(t)$ consists simply of the first $m$ components of $U^*(x(t))$

$$u(t) = [I \ 0 \ \dots \ 0] U^*(x(t)). \tag{8}$$

In Section 4, we will show that the solution $U^*(x)$ of the mp-QP problem is a continuous and piecewise affine function of $x$. Clearly, because of (8), the same properties are inherited by the controller.

Section 3 is devoted to investigate the regulation problem over an infinite prediction horizon $N_y = +\infty$, leading to solve explicitly the so-called *constrained linear quadratic regulation* (C-LQR) problem.

## 3. Piecewise linear solution to the constrained linear quadratic regulation problem

In their pioneering work, Sznaier and Damborg (1987), showed that finite horizon[3] optimization (3), (6), with $P$ satisfying (5), also provides the solution to the infinite-horizon linear quadratic regulation problem with constraints

$$V_\infty(x(0)) = \min_{u(0), u(1), \dots} \left\{ \sum_{t=0}^{\infty} x'(t) Q x(t) + u'(t) R u(t) \right\},$$
$$\begin{aligned} \text{s.t.} \quad & y_{\min} \leqslant Cx(t) \leqslant y_{\max}, \\ & u_{\min} \leqslant u(t) \leqslant u_{\max}, \\ & x(0) = x_0, \\ & x(t+1) = Ax(t) + Bu(t), \quad t \geqslant 0 \end{aligned} \tag{9}$$

with $Q$, $R$, $A$, and $B$ as in (3). The equivalence holds for a certain set of initial conditions, which depends on the length of the finite horizon. This idea has been reconsidered later by Chmielewski and Manousiouthakis (1996) independently by Scokaert and Rawlings (1998), and recently also in Chisci and Zappa (1999). In Scokaert and Rawlings (1998), the authors extend the idea of

[3] We refer to (3) as *finite horizon* also in the case $N_y = \infty$. In fact, such a problem can be transformed into a finite-horizon problem by choosing $P$ as in (4) or (5). Therefore "finiteness" should refer to the input and constraint horizons, as these are indeed the parameters which affect the complexity of the optimization problem (3).

Sznaier and Damborg (1987) by showing that the controller is stabilizing, and that the C-LQR problem is solved by a finite-dimensional QP problem. Unfortunately, the dimension of the QP depends on the initial state $x(0)$, and no upper-bound on the horizon (and therefore on the QP size) is given. On the other hand, Chmielewski and Manousiouthakis (1996) describe an algorithm which provides a *semi-global* upper-bound. Namely, for any given *compact set* $\chi_0$ of initial conditions, their algorithm provides the horizons $N_c = N_u$ such that the finite horizon controller (3), (6) solves the infinite horizon problem (9).

We briefly recall the approach given by Chmielewski and Manousiouthakis (1996) and propose some modifications, which will allow us to compute the closed-form of the constrained linear quadratic controller (9)

$$u(t) = f_\infty(x(t)), \quad \forall t \geqslant 0 \tag{10}$$

for a compact set of initial conditions $\chi(0)$, and show that it is piecewise affine.

Assume that (i) $Q \succ 0$, (ii) the sets $\chi$, $\mathscr{U}$ of feasible states and inputs, respectively, are bounded (e.g. $\|u_{\min}\|, \|u_{\max}\|, \|y_{\min}\|, \|y_{\max}\| < \infty$, and $C = I$) and contain the origin in their interior, and (iii) that for all the initial states $x(0) \in \chi(0)$, there exists an input sequence driving the system to the origin under constraints. We also assume that $\chi, \mathscr{U}$ are polytopes, as this is the case at hand in our paper, to simplify the exposition. Denote by $\mathscr{P}^N$, problem (3) with $N \triangleq N_u = N_c$, $N_y = \infty$, $K = K_{\mathrm{LQ}}$ (or, equivalently, $N_y = N$, and $P$ solving the Riccati equation (5b)) and by $\mathscr{P}_\infty$, problem (9). Note that in view of Bellman's principle of optimality (Bellman, 1961), (10) can be reformulated by substituting $x_{t+k|t}$, $u_{t+k}$ for $x(t)$, $u(t)$, respectively, as predicted and actual trajectories coincide.

Chmielewski and Manousiouthakis (1996) prove that, for a given $x(0)$, there exists a finite $N$ such that $\mathscr{P}^N$ and $\mathscr{P}^\infty$ are equivalent, and that the associated controller is exponentially stabilizing. The same result was proved in Scokaert and Rawlings (1998). In addition, by exploiting the convexity and continuity of the value function $V_\infty$ with respect to the initial condition $x(0)$, Chmielewski and Manousiouthakis (1996) provide the tools necessary to compute an upper-bound on $N$ for every given set $\chi(0)$ of initial conditions. Such an upper-bound is computed by modifying the algorithm in Chmielewski and Manousiouthakis (1996), as follows.

(1) Let $\bar{X} = \{x \in \chi\colon K_{\mathrm{LQ}}\, x \in \mathscr{U}\}$ the set of states such that the unconstrained LQ gain $K_{\mathrm{LQ}}$ is feasible. As $\chi, \mathscr{U}$ are polytopes, $\bar{X}$ is a polytope, defined by a set of linear inequalities of the form $\bar{A}^i x \leqslant \bar{b}^i$, $i = 1, \ldots, \bar{n}$, where $\bar{A}, \bar{b}, \bar{n}$ depend on the definition of $\chi, \mathscr{U}$.

(2) Compute $Z_c = \{x \in \mathbb{R}^n\colon x'Px \leqslant c\}$, where the largest $c$ such that $Z_c$ is an invariant subset of $\bar{X}$, is determined analytically (Bemporad, 1998) as

$$c = \min_{i=1,\ldots,\bar{n}} \frac{|\bar{b}^i|^2}{\bar{A}^i P^{-1} (\bar{A}^{i\prime})}. \tag{11}$$

(3) Let $q = c\lambda_{\min}(P^{-1}Q)$, where $\lambda_{\min}$ denotes the minimum eigenvalue.

(4) Let $\chi(0)$ be a compact set of initial conditions. Without loss of generality, assume that $\chi(0)$ is a polytope, and let $\{x_i\}_{i=1}^{\ell}$ be the set of its vertices. For each $x_i$ compute $V_\infty(x_i)$. This is done by computing the finite horizon value function $V(x_i)$ for increasing values of $N$ until $x_{t+k|N} \in Z_c$ (this computation is equivalent to the one described in Scokaert and Rawlings (1998), where a ball $B_r$ is used instead of $Z_c$).

(5) Let $U = \max_{i=1,\ldots,\ell} V_\infty(x_i)$, which is an upper bound to $V_\infty(x)$ on $\chi(0)$, as $V_\infty$ and $\chi(0)$ are convex (in Chmielewski & Manousiouthakis, 1996), a different approach is proposed where $U$ depends on $x_0$).

(6) Choose $N$ as the minimum integer such that $N \geqslant (U - c)/q$.

In Section 4, we will show that $\mathscr{P}^N$ has a continuous and piecewise affine solution. Therefore, on a compact set of initial conditions $\chi(0)$, the solution (10) to the CLQR problem $\mathscr{P}^\infty$ is also piecewise affine. Note that if $\chi_0 \supseteq \chi$ then the solution is also global because $\chi, \mathscr{U}$ are bounded.

## 4. Multi-parametric quadratic programming

In this section, we investigate multi-parametric quadratic programs (mp-QP) of form (7). We want to derive an algorithm to express the solution $U^*(x)$ and the minimum value $V(x) = J(U^*(x))$ as an explicit function of the parameters $x$, and to characterize the analytical properties of these functions. In particular, we will prove that the solution $U^*(x)$ is a continuous piecewise affine function of $x$, in the following sense.

**Definition 1.** A function $z(x)\colon X \mapsto \mathbb{R}^s$, where $X \subseteq \mathbb{R}^n$ is a polyhedral set, is piecewise affine if it is possible to partition $X$ into convex polyhedral regions, $CR_i$, and $z(x) = H^i x + k^i$, $\forall x \in CR_i$.

Piecewise quadraticity is defined analogously by letting $z(x)$ be a quadratic function $x'W^i x + H^i x + k^i$.

### 4.1. Fundamentals of the algorithm

Before proceeding further, it is useful to define

$$z \triangleq U + H^{-1}F'x(t), \tag{12}$$

$z \in \mathbb{R}^s$, and to transform (7) by completing squares to obtain the equivalent problem

$$V_z(x) = \min_z \frac{1}{2} z'Hz$$

$$\text{s.t.} \quad Gz \leqslant W + Sx(t), \tag{13}$$

where $S \triangleq E + GH^{-1}F'$, and $V_z(x) = V(x) - \frac{1}{2}x'(Y - FH^{-1}F')x$. In the transformed problem, the parameter vector $x$ appears only on the rhs of the constraints.

In order to start solving the mp-QP problem, we need an initial vector $x_0$ inside the polyhedral set $X = \{x\colon Tx \leqslant Z\}$ of parameters over which we want to solve the problem, such that the QP problem (13) is feasible for $x = x_0$. A good choice for $x_0$ is the center of the largest ball contained in $X$ for which a feasible $z$ exists, determined by solving the LP

$$\begin{aligned} \max_{x,z,\varepsilon} \quad & \varepsilon, \\ \text{s.t.} \quad & T^i x + \varepsilon \|T^i\| \leqslant Z^i, \\ & Gz - Sx \leqslant W \end{aligned} \tag{14}$$

(in particular, $x_0$ will be the Chebychev center of $X$ when the QP problem (13) is feasible for such an $x_0$). If $\varepsilon \leqslant 0$, then the QP problem (13) is infeasible for all $x$ in the interior of $X$. Otherwise, we fix $x = x_0$ and solve the QP problem (13), in order to obtain the corresponding optimal solution $z_0$. Such a solution is unique, because $H \succ 0$, and therefore uniquely determines a set of active constraints $\tilde{G}z_0 = \tilde{S}x_0 + \tilde{W}$ out of the constraints in (13). We can then prove the following result:

**Theorem 2.** *Let $H \succ 0$. Consider a combination of active constraints $\tilde{G}, \tilde{S}, \tilde{W}$, and assume that the rows of $\tilde{G}$ are linearly independent. Let $CR_0$ be the set of all vectors $x$, for which such a combination is active at the optimum ($CR_0$ is referred to as critical region). Then, the optimal $z$ and the associated vector of Lagrange multipliers $\lambda$ are uniquely defined affine functions of $x$ over $CR_0$.*

**Proof.** The first-order Karush–Kuhn–Tucker (KKT) optimality conditions (Bazaraa et al., 1993) for the mp-QP are given by

$$Hz + G'\lambda = 0, \quad \lambda \in \mathbb{R}^q, \tag{15a}$$

$$\lambda_i(G^i z - W^i - S^i x) = 0, \quad i = 1, \dots, q, \tag{15b}$$

$$\lambda \geqslant 0, \tag{15c}$$

$$Gz \leqslant W + Sx, \tag{15d}$$

where the superscript $i$ denotes the $i$th row. We solve (15a) for $z$,

$$z = -H^{-1}G'\lambda \tag{16}$$

and substitute the result into (15b) to obtain the complementary slackness condition $\lambda(-GH^{-1}G'\lambda - W - Sx) = 0$. Let $\breve{\lambda}$ and $\tilde{\lambda}$ denote the Lagrange multipliers corresponding to inactive and active constraints, respectively. For inactive constraints, $\breve{\lambda} = 0$. For active constraints, $-\tilde{G}H^{-1}\tilde{G}'\tilde{\lambda} - \tilde{W} - \tilde{S}x = 0$, and therefore,

$$\tilde{\lambda} = -(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + \tilde{S}x), \tag{17}$$

where $\tilde{G}, \tilde{W}, \tilde{S}$ correspond to the set of active constraints, and $(\tilde{G}H^{-1}\tilde{G}')^{-1}$ exists because the rows of $\tilde{G}$ are linearly independent. Thus, $\lambda$ is an affine function of $x$. We can substitute $\tilde{\lambda}$ from (17) into (16) to obtain

$$z = H^{-1}\tilde{G}'(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + \tilde{S}x) \tag{18}$$

and note that $z$ is also an affine function of $x$. $\square$

A similar result was obtained by Zafiriou (Chiou & Zafiriou, 1994; Zafiriou, 1990) based on the optimality conditions for QP problems reported in Fletcher (1981). Although his formulation is for finite impulse response models realized in a particular space form, where the state includes past inputs, his arguments can be adapted directly to the state-space formulation (3)–(6). However, his result does not make the piecewise linear dependence of $u$ on $x$ explicit, as the domains over which the different linear laws are defined are not characterized. We show next, that such domains are indeed polyhedral regions of the state space.

Theorem 2 characterizes the solution only locally in the neighborhood of a specific $x_0$, as it does not provide the construction of the set $CR_0$ where this characterization remains valid. On the other hand, this region can be characterized immediately. The variable $z$ from (16) must satisfy the constraints in (13):

$$GH^{-1}\tilde{G}'(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + \tilde{S}x) \leqslant W + Sx \tag{19}$$

and by (15c), the Lagrange multipliers in (17) must remain non-negative:

$$-(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + \tilde{S}x) \geqslant 0, \tag{20}$$

as we vary $x$. After removing the redundant inequalities from (19) and (20), we obtain a compact representation of $CR_0$. Obviously, $CR_0$ is a polyhedron in the $x$-space, and represents the largest set of $x \in X$ such that the combination of active constraints at the minimizer remains unchanged. Once the critical region $CR_0$ has been defined, the rest of the space $CR^{\text{rest}} = X - CR_0$ has to be explored and new critical regions generated. An effective approach for partitioning the rest of the space was proposed in Dua and Pistikopoulos (2000). The following theorem justifies such a procedure to characterize the rest of the region $CR^{\text{rest}}$.

**Theorem 3.** *Let* $Y \subseteq \mathbb{R}^n$ *be a polyhedron, and* $CR_0 \triangleq \{x \in Y\colon Ax \leqslant b\}$ *a polyhedral subset of* $Y$, $CR_0 \neq \emptyset$. *Also let*

$$R_i = \left\{ x \in Y\colon \begin{array}{l} A^i x > b^i \\ A^j x \leqslant b^j, \ \forall j < i \end{array} \right\}, \quad i = 1, \ldots, m,$$

*where* $m = \dim(b)$, *and let* $CR^{\text{rest}} \triangleq \bigcup_{i=1}^{m} R_i$. *Then* (i) $CR^{\text{rest}} \cup CR_0 = Y$, (ii) $CR_0 \cap R_i = \emptyset$, $R_i \cap R_j = \emptyset$, $\forall i \neq j$, *i.e.* $\{CR_0, R_1, \ldots, R_m\}$ *is a partition of* $Y$.

**Proof.** (i) We want to prove that given an $x \in Y$, $x$ either belongs to $CR_0$ or to $R_i$ for some $i$. If $x \in CR_0$, we are done. Otherwise, there exists an index $i$ such that $A^i x > b^i$. Let $i^* = \min_{i \leqslant m}\{i\colon A^i x > b^i\}$. Then, $x \in R_{i^*}$, because $A^{i^*} x > b^{i^*}$ and $A^j x \leqslant b^j$, $\forall j < i^*$, by definition of $i^*$.

(ii) Let $x \in CR_0$. Then $\nexists i$ such that $A^i x > b^i$, which implies that $x \notin R_i$, $\forall i \leqslant m$. Let $x \in R_i$ and take $i > j$. Since $x \in R_i$, by definition of $R_i$ $(i > j)$ $A^j x \leqslant b^j$, which implies that $x \notin R_j$. □

**Example 4.1.** In order to exemplify the procedure proposed in Theorem 3 for partitioning the set of parameters $X$, consider the case when only two parameters $x_1$ and $x_2$ are present. As shown in Fig. 1(a), $X$ is defined by the inequalities $\{x_1^- \leqslant x_1 \leqslant x_1^+,\ x_2^- \leqslant x_2 \leqslant x_2^+\}$, and $CR_0$ by the inequalities $\{C1 \leqslant 0, \ldots, C5 \leqslant 0\}$ where $C1, \ldots, C5$ are affine functions of $x$. The procedure consists of considering, one by one, the inequalities which define $CR_0$. Considering, for example, the inequality $C1 \leqslant 0$, the first set of the rest of the region $CR^{\text{rest}} \triangleq X \backslash CR_0$ is given by $R_1 = \{C1 \geqslant 0,\ x_1 \geqslant x_1^-,\ x_2^- \leqslant x_2 \leqslant x_2^+\}$, which is obtained by reversing the sign of the inequality $C1 \leqslant 0$ and removing redundant constraints (see Fig. 1(b)). Thus, by considering the rest of the inequalities, the complete rest of the region is $CR^{\text{rest}} = \bigcup_{i=1}^{5} R_i$, where $R_1, \ldots, R_5$ are graphically reported in Fig. 1(d). Note that the partition strategy suggested by Theorem 3 can be also applied also when $X$ is unbounded.

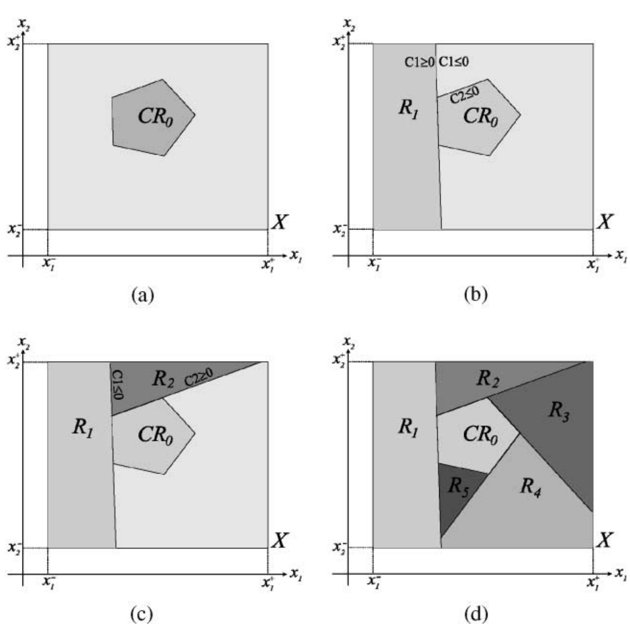


Fig. 1. Partition of $CR^{\text{rest}} \triangleq X \backslash CR_0$; (b) partition of $CR^{\text{rest}}$ step 1; (c) partition of $CR^{\text{rest}}$ Step 2; and (d) final partition of $CR^{\text{rest}}$.

Theorem 3 provides a way of partitioning the non-convex set, $X \backslash CR_0$, into polyhedral subsets $R_i$. For each $R_i$, a new vector $x_i$ is determined by solving the LP (14), and, correspondingly, an optimum $z_i$, a set of active constraints $\tilde{G}^i, \tilde{S}^i, \tilde{W}^i$, and a critical region $CR_i$. Theorem 3 is then applied to partition $R_i \backslash CR_i$ into polyhedral subsets, and the algorithm proceeds iteratively. The complexity of the algorithm will be fully discussed in Section 4.3.

Note that Theorem 3 introduces cuts in the $x$-space which might split critical regions into subsets. Therefore, after the whole $x$-space has been covered, those polyhedral regions $CR^i$ are determined where the function $z(x)$ is the same. If their union is a convex set, it is computed to permit a more compact description of the solution (Bemporad, Fukuda, & Torrisi, 2001). Alternatively, in Borrelli, Bemporad, and Morari (2000), the authors propose not to intersect (19)–(20) with the partition generated by Theorem 3, and simply use Theorem 3 to guide the exploration. As a result, some critical regions may appear more than once. Duplicates can be easily eliminated by recognizing regions where the combination of active constraints is the same. In the sequel, we will denote by $N_{\text{r}}$, the final number of polyhedral cells defining the mp-QP solution (i.e., after the union of neighboring cells or removal of duplicates, respectively).

### 4.1.1. Degeneracy

So far, we have assumed that the rows of $\tilde{G}$ are linearly independent. It can happen, however, that by solving the QP (13), one determines a set of active constraints for which this assumption is violated. For instance, this happens when more than $s$ constraints are active at the optimizer $z_0 \in \mathbb{R}^s$, i.e., in a case of *primal degeneracy*. In this case, the vector of Lagrange multipliers $\lambda_0$ might not be uniquely defined, as the dual problem of (13) is not strictly convex (instead, *dual degeneracy* and non-uniqueness of $z_0$ cannot occur, as $H \succ 0$). Let $\tilde{G} \in \mathbb{R}^{\ell \times s}$, and let $r = \operatorname{rank} \tilde{G}$, $r < \ell$. In order to characterize such a degenerate situation, consider the QR decomposition $\tilde{G} = \left[\begin{smallmatrix} R_1 \\ 0 \end{smallmatrix}\right] Q$ of $\tilde{G}$, and rewrite the active constraints in the form

$$R_1 z = W_1 + S_1 x, \tag{21a}$$

$$0 = W_2 + S_2 x, \tag{21b}$$

where $\left[\begin{smallmatrix} S_1 \\ S_2 \end{smallmatrix}\right] = Q^{-1}\tilde{S}$, $\left[\begin{smallmatrix} W_1 \\ W_2 \end{smallmatrix}\right] = Q^{-1}\tilde{W}$. If $S_2$ is non-zero, because of the equalities (21b), $CR_0$ is a lower-dimensional region, which, in general, corresponds to a common boundary between two or more full-dimensional regions.[4] Therefore, it is not worth to explore this combination $\tilde{G}, \tilde{S}, \tilde{W}$. On the other hand, if both $W_2$ and $S_2$ are zero, the KKT conditions do not lead directly to (19) and (20), but only to a polyhedron expressed in the $(\lambda, x)$ space. In this case, a full-dimensional critical region can be obtained by a projection algorithm (Fukuda, 1997), which, however, is computationally expensive (the case $W_2 \neq 0$, $S_2 = 0$ is not possible, since the LP (14) was feasible).

In this paper, we suggest a simpler way to handle such a degenerate situation, which consists of collecting $r$ constraints arbitrarily chosen, and proceed with the new reduced set, therefore avoiding the computation of projections. Due to the recursive nature of the algorithm, the remaining other possible subsets of combinations of constraints leading to full-dimensional critical regions will automatically be explored later.

### 4.2. Continuity and convexity properties

Continuity of the value function $V_z(x)$ and the solution $z(x)$, can be shown as simple corollaries of the linearity result of Theorem 2. This fact, together with the convexity of the set of feasible parameters $X_f \subseteq X$ (i.e. the set of parameters $x \in X$ such that a feasible solution $z(x)$ exists to the optimization problem (13)), and of the value function $V_z(x)$, is proved in next Theorem.

**Theorem 4.** *Consider the multi-parametric quadratic program* (13) *and let* $H \succ 0$, $X$ *convex. Then the set of feasible parameters* $X_f \subseteq X$ *is convex, the optimizer* $z(x): X_f \mapsto \mathbb{R}^s$ *is continuous and piecewise affine, and the optimal solution* $V_z(x): X_f \mapsto \mathbb{R}$ *is continuous, convex and piecewise quadratic.*

**Proof.** We first prove convexity of $X_f$ and $V_z(x)$. Take generic $x_1, x_2 \in X_f$, and let $V_z(x_1)$, $V_z(x_2)$ and $z_1, z_2$ the corresponding optimal values and minimizers. Let $\alpha \in [0,1]$, and define $z_\alpha \triangleq \alpha z_1 + (1-\alpha) z_2$, $x_\alpha \triangleq \alpha x_1 + (1-\alpha) x_2$. By feasibility, $z_1, z_2$ satisfy the constraints $Gz_1 \leqslant W + Sx_1$, $Gz_2 \leqslant W + Sx_2$. These inequalities can be linearly combined to obtain $Gz_\alpha \leqslant W + Sx_\alpha$, and therefore, $z_\alpha$ is feasible for the optimization problem (13), where $x(t) = x_\alpha$. This proves that $z(x_\alpha)$ exists, and therefore, convexity of $X_f = \bigcup_i CR_i$. In particular, $X_f$ is connected. Moreover, by optimality of $V_z(x_\alpha)$, $V_z(x_\alpha) \leqslant \frac{1}{2} z_\alpha' H z_\alpha$, and hence $V_z(x_\alpha) - \frac{1}{2}[\alpha z_1' H z_1 + (1-\alpha) z_2' H z_2] \leqslant \frac{1}{2} z_\alpha' H z_\alpha - \frac{1}{2}[\alpha z_1' H z_1 + (1-\alpha) z_2' H z_2] = \frac{1}{2}[\alpha^2 z_1' H z_1 + (1-\alpha)^2 z_2' H z_2 + 2\alpha(1-\alpha) z_2' H z_1 - \alpha z_1' H z_1 - (1-\alpha) z_2' H z_2] = -\frac{1}{2}\alpha(1-\alpha)(z_1 - z_2)' H (z_1 - z_2) \leqslant 0$, i.e. $V_z(\alpha x_1 + (1-\alpha) x_2) \leqslant \alpha V_z(x_1) + (1-\alpha) V_z(x_2)$, $\forall x_1, x_2 \in X$, $\forall \alpha \in [0,1]$, which proves the convexity of $V_z(x)$ on $X_f$. Within the closed polyhedral regions $CR_i$ in $X_f$, the solution $z(x)$ is affine (18). The boundary between two regions belongs to both closed regions. Since $H \succ 0$, the optimum is unique, and hence, the solution must be continuous across the boundary. The fact that $V_z(x)$ is continuous and piecewise quadratic, follows trivially. □

In order to prove the convexity of the value function $V(x)$ of the MPC problems (3) and (7), we need the following lemma.

**Lemma 1.** *Let* $J(U,x) = \frac{1}{2} U' H U + x' F U + \frac{1}{2} x' Y x$, *and let* $\begin{bmatrix} Y & F' \\ F & H \end{bmatrix} \succcurlyeq 0$. *Then* $V(x) \triangleq \min_U J(x,U)$ *subject to* $GU \leqslant W + Ex$ *is a convex function of* $x$.

**Proof.** By Theorem 4, the value function $V_z(x)$ of the optimization problem $V_z(x) \triangleq \min_z \frac{1}{2} z' H z$ subject to $Gz \leqslant W + Sx$ is a convex function of $x$. Let $z^*(x)$, $U^*(x)$ be the optimizers of $V_z(x)$ and $V(x)$, respectively, where $z^*(x) = U^*(x) + H^{-1} F x$. Then, $V(x) = \frac{1}{2} U^*(x) H U^*(x) + x' F U^*(x) + \frac{1}{2} x' Y x = \frac{1}{2}[z^*(x) - H^{-1} F x]' H [z^*(x) - H^{-1} F x] + x' F [z^*(x) - H^{-1} F x] + \frac{1}{2} x' Y x = V_z(x) - \frac{1}{2} x' F H^{-1} F' x + \frac{1}{2} x' Y x = V_z(x) + \frac{1}{2} x'(Y - F H^{-1} F') x$. As $\begin{bmatrix} Y & F' \\ F & H \end{bmatrix} \succcurlyeq 0$, its Schur's complement $Y - F H^{-1} F' \succcurlyeq 0$, and therefore, $V(x)$ is a convex function, being the sum of convex functions. □

**Corollary 1.** *The value function* $V(x)$ *defined by the optimization problem* (3), (7) *is continuous and piecewise quadratic.*

**Proof.** By (3), $J(U,x) \geqslant 0$, $\forall x, U$, being the sum of non-negative terms. Therefore, $J(U,x) = \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]' \begin{bmatrix} Y & F' \\ F & H \end{bmatrix} \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \geqslant 0$ for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$, and the proof easily follows by Lemma 1. □

A simple consequence of Corollary 1 is that the Lyapunov function used to prove Theorem 1 is continuous, convex, and piecewise quadratic. Finally, we can establish the analytical properties of the controller (3), (6) through the following corollary of Theorem 4.

[4] In general, the set of feasible parameters $x$ can be a lower-dimensional subset of $X$ (Filippi, 1997). However, when in the MPC formulation (3), $u_{\min} < u_{\max}$, $y_{\min} < y_{\max}$, the mp-QP problem has always a full-dimensional solution in the $x$-space, as the critical region corresponding to the unconstrained solution contains a full-dimensional ball around the origin.

**Corollary 2.** *The control law* $u(t) = f(x(t))$, $f: \mathbb{R}^n \mapsto \mathbb{R}^m$, *defined by the optimization problem* (3) *and* (6) *is continuous and piece-wise affine*

$$f(x) = F^i x + g^i \quad \text{if } H^i x \leqslant k^i, \; i = 1, \ldots, N_{\text{mpc}} \tag{22}$$

*where the polyhedral sets* $\{H^i x \leqslant k^i\}$, $i = 1, \ldots, N_{\text{mpc}} \leqslant N_{\text{r}}$ *are a partition of the given set of states* $X$.

**Proof.** By (12), $U(x) = z(x) - H^{-1}F'x$ is a linear function of $x$ in each region $CR_i = \{x\colon H^i x \leqslant K^i, i = 1, \ldots, N_{\text{mpc}}\}$. By (8), $u = f(x)$ is a combination of linear functions, and therefore, is linear on $CR_i$. Also, $u$ is a combination of continuous functions, and therefore, is continuous. □

Multiparametric quadratic programming problems can also be addressed by employing the principles of parametric non-linear programming, exploiting the Basic Sensitivity Theorem (Fiacco, 1976, 1983) (a direct consequence of the KKT conditions and the Implicit Function Theorem). In this paper, we opted for a more direct approach, which exploits the linearity of the constraints and the fact that the function to be minimized is quadratic.

We remark that the command actions provided by the (explicit) feedback control law (22) and the (implicit) feedback control law (3)–(6) are *exactly equal*. Therefore, the control designer is allowed to tune the controller by using standard MPC tools (i.e., based on on-line QP solvers) and software simulation packages, and finally run Algorithm 1 to obtain the explicit solution (22) to efficiently implement the controller.

### 4.3. Off-line algorithm for mp-QP and explicit MPC

Based on the above discussion and results, the main steps of the off-line mp-QP solver are outlined in the following algorithm.

**Algorithm 1**

1 **Let** $X \subseteq \mathbb{R}^n$ be the set of parameters (states);
2 execute **partition**($X$);
3 for all regions where $z(x)$ is the same and whose union is a convex set, compute such a union as described by Bemporad, Fukuda, and Torrisi (2001);
4 **end**.
**procedure partition**($Y$)
1.1 **let** $x_0 \in Y$ and $\varepsilon$ the solution to the LP (14);
1.2 **if** $\varepsilon \leqslant 0$ **then exit**; (no full dimensional CR is in $Y$)
1.3 For $x = x_0$, compute the optimal solution $(z_0, \lambda_0)$ of the QP (13);
1.4 Determine the set of active constraints when $z = z_0$, $x = x_0$, and build $\tilde{G}, \tilde{W}, \tilde{S}$;
1.5 If $r = \operatorname{rank} \tilde{G}$ is less than the number $\ell$ of rows of $\tilde{G}$, take a subset of $r$ linearly independent rows, and redefine $\tilde{G}, \tilde{W}, \tilde{S}$ accordingly;
1.6 Determine $\tilde{\lambda}(x)$, $z(x)$ from (17) and (18);
1.7 Characterize the CR from (19) and (20);
1.8 Define and partition the rest of the region as in Theorem 3;
1.9 For each new sub-region $R_i$, **partition**($R_i$);
**end procedure**.

The algorithm explores the set $X$ of parameters recursively: Partition the rest of the region as in Theorem 3 into polyhedral sets $R_i$, use the same method to partition each set $R_i$ further, and so on. This can be represented as a search tree, with a maximum depth equal to the number of combinations of active constraints (see Section 4.4 below).

The algorithm solves the mp-QP problem by partitioning the given parameter set $X$ into $N_{\text{r}}$ convex polyhedral regions. For the characterization of the MPC controller, in step 3 the union of regions is computed where the first $N_u$ components of the solution $z(x)$ are the same, by using the algorithm developed by Bemporad, Fukuda, and Torrisi (2001). This reduces the total number of regions in the partition for the MPC controller from $N_{\text{r}}$ to $N_{\text{mpc}}$.

### 4.4. Complexity analysis

The number $N_{\text{r}}$ of regions in the mp-QP solution depends on the dimension $n$ of the state, and on the number of degrees of freedom $s = mN_u$ and constraints $q$ in the optimization problem (13). As the number of combinations of $\ell$ constraints out of a set of $q$ is $\binom{q}{\ell} = q!/(q-\ell)!\ell!$, the number of possible combinations of active constraints at the solution of a QP is at most $\sum_{\ell=0}^{q} \binom{q}{\ell} = 2^q$. This number represents an upper bound on the number of different linear feedback gains which describe the controller. In practice, far fewer combinations are usually generated as $x$ spans $X$. Furthermore, the gains for the future input moves $u_{t+1}, \ldots, u_{t+N_u-1}$ are not relevant for the control law. Thus, several different combinations of active constraints may lead to the same first $m$ components $u_t^*(x)$ of the solution. On the other hand, the number $N_{\text{r}}$ of regions of the piecewise affine solution is, in general, larger than the number of feedback gains, because non-convex critical regions are split into several convex sets. For instance, the example reported in Fig. 6(d) involves 13 feedback gains, distributed over 57 regions of the state space.

A worst-case estimate of $N_{\text{r}}$ can be computed from the way Algorithm 1 generates critical regions. The first critical region $CR_0$ is defined by the constraints $\lambda(x) \geqslant 0$ ($q$ constraints) and $Gz(x) \leqslant W + Sx$ ($q$ constraints). If the strict complementary slackness condition holds, only

$q$ constraints can be active, and hence, every $CR$ is defined by $q$ constraints. From Theorem 3, $CR^{\text{rest}}$ consists of $q$ convex polyhedra $R_i$, defined by at most $q$ inequalities. For each $R_i$, a new $CR_i$ is determined which consists of $2q$ inequalities (the additional $q$ inequalities come from the condition $CR_i \subseteq R_i$), and therefore, the corresponding $CR^{\text{rest}}$ partition includes $2q$ sets defined by $2q$ inequalities. As mentioned above, this way of generating regions can be associated with a search tree. By induction, it is easy to prove that at the tree level $k+1$, there are $k!m^k$ regions defined by $(k+1)q$ constraints. As observed earlier, each $CR$ is the largest set corresponding to a certain combination of active constraints. Therefore, the search tree has a maximum depth of $2^q$, as at each level, there is one admissible combination less. In conclusion, the number of regions in the solution to the mp-QP problem is $N_{\text{r}} \leqslant \sum_{k=0}^{2^q-1} k!q^k$, each one defined by at most $q2^q$ linear inequalities. Note that the above analysis is largely overestimating the complexity, as it does not take into account: (*i*) the elimination of redundant constraints when a CR is generated, and (*ii*) that empty sets are not partitioned further.

### 4.5. Dependence on $n, m, p, N_u, N_{\text{c}}$

Let $q_s \triangleq \text{rank}\, S$, $q_s \leqslant q$. For $n > q_s$ the number of polyhedral regions $N_{\text{r}}$ remains constant. To see this, consider the linear transformation $\bar{x} = Sx$, $\bar{x} \in \mathbb{R}^q$. Clearly, $\bar{x}$ and $x$ define the same set of active constraints, and therefore the number of partitions in the $\bar{x}$- and $x$-space are the same. Therefore, the number of partitions, $N_{\text{r}}$, of the $x$-space defining the optimal controller is insensitive to the dimension $n$ of the state $x$ for all $n \geqslant q_s$, i.e. to the number of parameters involved in the mp-QP. In particular, the additional parameters that we will introduce in Section 6 to extend MPC to reference tracking, disturbance rejection, soft constraints, variable constraints, and output feedback, do not affect significantly, the number of polyhedral regions $N_{\text{r}}$ (i.e., the complexity of the mp-QP), and hence, the number $N_{\text{mpc}}$ of regions in the MPC controller (22).

The number $q$ of constraints increases with $N_{\text{c}}$ and, in the case of input constraints, with $N_u$. For instance, $q = 2s = 2mN_u$ for control problems with input constraints only. From the analysis above, the larger $N_{\text{c}}, N_u, m, p$, the larger $q$, and therefore $N_{\text{r}}$. Note that many control problems involve input constraints only, and typically horizons $N_u = 2,3$ or blocking of control moves are adopted, which reduces the number of constraints $q$.

### 4.6. Off-line computation time

In Table 1, we report the computation time and the number of regions obtained by solving a few test MPC problems on random SISO plants subject to input constraints. In the comparison, we vary the number of free moves $N_u$ and the number of states of the open-loop system. Computation times were evaluated by running Algorithm 1 in Matlab 5.3 on a Pentium III-650 MHz machine. No attempts were made to optimize the efficiency of the algorithm and its implementation.

Table 1
Off-line computation time to solve the mp-QP problem and, in parentheses, number of regions $N_{\text{mpc}}$ in the MPC controller ($N_u$ = number of control moves, $n$ = number of states)

| $N_u$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ |
|---|---|---|---|---|
| 2 | 0.44 s (7) | 0.49 s (7) | 0.55 s (7) | 1.43 s (7) |
| 3 | 1.15 s (13) | 2.08 s (15) | 1.75 s (15) | 5.06 s (17) |
| 4 | 2.31 s (21) | 5.87 s (29) | 3.68 s (29) | 15.93 s (37) |

### 4.7. On-line computation time

The simplest way to implement the piecewise affine feedback law (22) is to store the polyhedral cells $\{H^i x \leqslant k^i\}$, perform an on-line linear search through them to locate the one which contains $x(t)$, then lookup the corresponding feedback gain $(F^i, g^i)$. This procedure can be easily parallelized (while for a QP solver, the parallelization is less obvious). However, more efficient on-line implementation techniques which avoid the storage and the evaluation of the polyhedral cells are currently under development.

## 5. State-feedback solution to constrained linear quadratic control

For $t = 0$, the explicit solution to (3) provides the optimal input profile $u(0), \ldots, u(N_y - 1)$ as a function of the state $x(0)$. The equivalent state-feedback form $u(j) = f_j(x(j))$, $j = 0, \ldots, N_y - 1$ can be obtained by solving $N_u$ mp-QPs. In fact, clearly, $f_j(x) = Kx$ for all $j = N_u, \ldots, N_y - 1$, where $f_j : X_j \mapsto \mathbb{R}^m$, and $X_j \triangleq \mathbb{R}^n$ for $j = N_{\text{c}} + 1, \ldots, N_y$, $X_j \triangleq \{x\colon y_{\min} \leqslant C(A+BK)^h x \leqslant y_{\max}$, $u_{\min} \leqslant K(A+BK)^h x \leqslant u_{\max}$, $h = 0, \ldots, N_{\text{c}} - j\}$ for $j = N_u, \ldots, N_{\text{c}}$. For $0 \leqslant j \leqslant N_u - 1$, $f_j(x)$ is obtained by solving the mp-QP problem

$$F_j(x) \triangleq \min_{U \triangleq \{u(j), \ldots, u(N_u - 1)\}} \Big\{ J(U, x) = x'(N_y) P x(N_y) + \sum_{k=j}^{N_y - 1} [x'(k) Q x(k) + u'(k) R u(k)] \Big\}$$

$$\begin{aligned}
\text{s.t.} \quad & y_{\min} \leqslant y(k) \leqslant y_{\max}, \quad k = j, \dots, N_c, \\
& u_{\min} \leqslant u(k) \leqslant u_{\max}, \quad k = j, \dots, N_c, \\
& x(j) = x, \\
& x(k+1) = Ax(k) + Bu(k), \quad k \geqslant j, \\
& y(k) = Cx(k), \quad k \geqslant j, \\
& u(k) = Kx(k), \quad N_u \leqslant k < N_y
\end{aligned} \tag{23}$$

and setting $f_j(x) = [I \ 0 \ \dots \ 0]F_j(x)$ (note that, compared to (3), for $j = 0$, (23) includes the extra constraint $y_{\min} \leqslant y(0) \leqslant y_{\max}$. However, this may only restrict the set of $x(0)$ for which (23) is feasible, but does not change the control function $f_0(x)$, as $u(0)$ does not affect $y(0)$).

Similar to the unconstrained finite-time LQ problem, where the state-feedback solution is linear time-varying, the explicit state-feedback solution to (3) is the time-varying piecewise affine function $f_j: \mathbb{R}^m \mapsto \mathbb{R}^n$, $j = 0, \dots, N_y - 1$. Note that while in the unconstrained case dynamic programming nicely provides the state-feedback solution through Riccati iterations, because of the constraints here, dynamic programming would lead to solving a sequence of multiparametric *piecewise* quadratic programs, instead of the mp-QPs (23).

The infinite horizon-constrained linear quadratic regulator can also be obtained in state-feedback form by choosing $N_u = N_y = N_c = N$, where $N$ is defined according to the results of Section 3.

## 6. Reference tracking, disturbances, and other extensions

The basic formulation (3) can be extended naturally to situations where the control task is more demanding. As long as the control task can be expressed as an mp-QP, a piecewise affine controller results, which can be easily designed and implemented. In this section, we will mention only a few extensions to illustrate the potential. To our knowledge, these types of problems are difficult to formulate from the point of view of anti-windup or other techniques not related to MPC.

### 6.1. Reference tracking

The controller can be extended to provide offset-free tracking of asymptotically constant reference signals. Future values of the reference trajectory can be taken into account explicitly, by the controller, so that the control action is optimal for the future trajectory in the presence of constraints.

Let the goal be to have the output vector $y(t)$ track $r(t)$, where $r(t) \in \mathbb{R}^p$ is the reference signal. To this aim, consider the MPC problem

$$\begin{aligned}
\min_{U \triangleq \{\delta u_t, \dots, \delta u_{t+N_u-1}\}} & \sum_{k=0}^{N_y-1} \{[y_{t+k|t} - r(t)]' Q [y_{t+k|t} - r(t)] \\
& + \delta u'_{t+k|t} R \delta u_{t+k|t}\} \\
\text{s.t.} \quad & y_{\min} \leqslant y_{t+k|t} \leqslant y_{\max}, \quad k = 1, \dots, N_c, \\
& u_{\min} \leqslant u_{t+k} \leqslant u_{\max}, \quad k = 0, 1, \dots, N_c, \\
& \delta u_{\min} \leqslant \delta u_{t+k} \leqslant \delta u_{\max}, \quad k = 0, 1, \dots, N_u - 1, \\
& x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k}, \quad k \geqslant 0, \\
& y_{t+k|t} = Cx_{t+k|t}, \quad k \geqslant 0, \\
& u_{t+k} = u_{t+k-1} + \delta u_{t+k}, \quad k \geqslant 0, \\
& \delta u_{t+k} = 0, \quad k \geqslant N_u.
\end{aligned} \tag{24}$$

Note that the $\delta u$-formulation (24) introduces $m$ new states in the predictive model, namely, the last input $u(t-1)$ (this corresponds to adding an integrator in the control loop). Just like the regulation problem (3), we can transform the tracking problem (24) into the form

$$\begin{aligned}
\min_U \ & \frac{1}{2} U' H U + [x'(t) \ u'(t-1) \ r'(t)] F U \\
\text{s.t.} \quad & GU \leqslant W + E \begin{bmatrix} x(t) \\ u(t-1) \\ r(t) \end{bmatrix},
\end{aligned} \tag{25}$$

where $r(t)$ lies in a given (possibly unbounded) polyhedral set. Thus, the same mp-QP algorithm can be used to obtain an explicit piecewise affine solution $\delta u(t) = F(x(t), u(t-1), r(t))$. In case the reference $r(t)$ is known in advance, one can replace $r(t)$ by $r(t+k)$ in (24) and similarly get a piecewise affine anticipative controller $\delta u(t) = F(x(t), u(t-1), r(t), \dots, r(t+N_y-1))$.

### 6.2. Disturbances

We distinguish between *measured* and *unmeasured* disturbances. Measured disturbances $v(t)$ can be included in the prediction model

$$x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k} + Vv(t+k|t), \tag{26}$$

where $v(t+k|t)$ is the prediction of the disturbance at time $t+k$ based on the measured value $v(t)$. Usually, $v(t+k|t)$ is a linear function of $v(t)$, for instance $v(t+k|t) \equiv v(t)$ where it is assumed that the disturbance is constant over the prediction horizon. Then $v(t)$ appears as a vector of additional parameters in the mp-QP, and the piecewise affine control law becomes $u(t) = F(x(t), v(t))$. Alternatively, as for reference tracking, when $v(t)$ is known in advance one can replace $v(t+k|t)$ by $v(t+k)$ in (26) and get an anticipative controller $\delta u(t) = F(x(t), u(t-1), v(t), \dots, v(t+N_y-1))$.

Usually unmeasured disturbances are modeled as the output of a linear system driven by a white Gaussian noise. The state vector $x(t)$ of the linear prediction model (1) is augmented by the state $x_n(t)$ of such a linear disturbance model, and the mp-QP provides a control law of the form $\delta u(t) = F(x(t), x_n(t))$ within a certain range of states of the plant and of the disturbance model. Clearly, $x_n(t)$ is estimated on line from output measurements by a linear observer.

### 6.3. Soft constraints

State and output constraints can lead to feasibility problems. For example, a disturbance may push the output outside the feasible region where no allowed control input may exist which brings the output back inside at the next time step. Therefore, in practice, the output constraints (2) are relaxed or softened (Zheng & Morari, 1995) as $y_{\min} - M\varepsilon \leqslant y(t) \leqslant y_{\max} + M\varepsilon$, where $M \in \mathbb{R}^p$ is a constant vector ($M^i \geqslant 0$ is related to the "concern" for the violation of the $i$th output constraint), and the term $\rho\varepsilon^2$ is added to the objective to penalize constraint violations ($\rho$ is a suitably large scalar). The variable $\varepsilon$ plays the role of an independent optimization variable in the mp-QP and is adjoined to $z$. The solution $u(t) = F(x(t))$ is again a piecewise affine controller, which aims at keeping the states in the constrained region without ever running into feasibility problems.

### 6.4. Variable constraints

The bounds $y_{\min}, y_{\max}, \delta u_{\min}, \delta u_{\max}, u_{\min}, u_{\max}$ may change depending on the operating conditions, or in the case of a stuck actuator, the constraints become $\delta u_{\min} = \delta u_{\max} = 0$. This possibility can again be built into the control law. The bounds can be treated as parameters in the mp-QP and added to the vector $x$. The control law will have the form $u(t) = F(x(t), y_{\min}, y_{\max}, \delta u_{\min}, \delta u_{\max}, u_{\min}, u_{\max})$.

## 7. Examples

### 7.1. A simple SISO system

Consider the second order system

$$y(t) = \frac{2}{s^2 + 3s + 2}\, u(t),$$

sample the dynamics with $T = 0.1$ s, and obtain the state-space representation

$$x(t+1) = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} x(t) + \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix} u(t).$$

$$y(t) = [0 \;\; 1.4142]\, x(t). \tag{27}$$

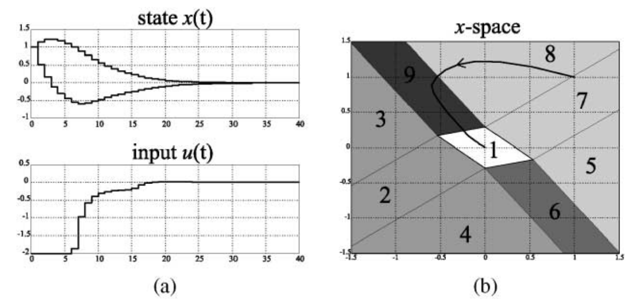


Fig. 2. Example 7.1: (a) closed-loop MPC; (b) state-space partition and closed-loop MPC trajectories.

The task is to regulate the system to the origin while fulfilling the input constraint

$$-2 \leqslant u(t) \leqslant 2. \tag{28}$$

To this aim, we design an MPC controller based on the optimization problem

$$\begin{aligned} \min_{u_t, u_{t+1}} \quad & x'_{t+2|t} P x_{t+2|t} + \sum_{k=0}^{1} [x'_{t+k|t} x_{t+k|t} + 0.01 u^2_{t+k}] \\ \text{s.t.} \quad & -2 \leqslant u_{t+k} \leqslant 2, \quad k = 0,1, \\ & x_{t|t} = x(t) \end{aligned} \tag{29}$$

where $P$ solves the Lyapunov equation $P = A'PA + Q$ (in this example, $Q = I$, $R = 0.01$, $N_y = N_u = 2$, $N_c = 1$). Note that this choice of $P$ corresponds to setting $u_{t+k} = 0$ for $k \geqslant 2$ and minimize

$$\sum_{k=0}^{\infty} x'_{t+k|t} x_{t+k|t} + 0.01 u^2_{t+k}. \tag{30}$$

The MPC controller (29) is globally asymptotically stabilizing. In fact, it is easy to show that the value function is a Lyapunov function of the system.[5] The closed-loop response from the initial condition $x(0) = [1 \;\; 1]'$ is shown in Fig. 2(a).

The mp-QP problem associated with the MPC law has the form (7) with

$$H = \begin{bmatrix} 1.5064 & 0.4838 \\ 0.4838 & 1.5258 \end{bmatrix}, \qquad F = \begin{bmatrix} 9.6652 & 5.2115 \\ 7.0732 & -7.0879 \end{bmatrix},$$

[5] Let $U_t^* = [u_1^*, u_2^*]'$ be the optimal solution at time $t$. Then $U = [u_2^*, 0]'$ is feasible at time $t+1$. The cost associated with $U$ is $J(t+1, U) = J(t, U_t^*) - x'(t)x(t) - 0.01u^2(t) \geqslant J(t+1, U_{t+1}^*)$, which implies that $J(t, U_t^*)$ is a converging sequence. Therefore, $x'(t)x(t) + 0.01u^2(t) \leqslant J(t, U_t^*) - J(t+1, U_{t+1}^*) \to 0$, which shows stability of the system.

$$G = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad W = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The solution was computed by Algorithm 1 in 0.66 s (15 regions examined), and the corresponding polyhedral partition of the state-space into $N_r = 9$ polyhedral cells is depicted in Fig. 2(b). The MPC law is

$$u = \begin{cases}
[-5.9220 \;\; -6.8883]x & \text{if} \begin{bmatrix} -5.9220 & -6.8883 \\ 5.9220 & 6.8883 \\ -1.5379 & 6.8291 \\ 1.5379 & -6.8291 \end{bmatrix} x \leqslant \begin{bmatrix} 2.0000 \\ 2.0000 \\ 2.0000 \\ 2.0000 \end{bmatrix}, \\
& \text{(Region \#1)} \\
2.0000 & \text{if} \begin{bmatrix} -3.4155 & 4.6452 \\ 0.1044 & 0.1215 \\ 0.1259 & 0.0922 \end{bmatrix} x \leqslant \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix}, \\
& \text{(Region \#2, \#4)} \\
2.0000 & \text{if} \begin{bmatrix} 0.0679 & -0.0924 \\ 0.1259 & 0.0922 \end{bmatrix} x \leqslant \begin{bmatrix} -0.0524 \\ -0.0519 \end{bmatrix}, \\
& \text{(Region \#3)} \\
-2.0000 & \text{if} \begin{bmatrix} -0.1259 & -0.0922 \\ -0.0679 & 0.0924 \end{bmatrix} x \leqslant \begin{bmatrix} -0.0519 \\ -0.0524 \end{bmatrix}, \\
& \text{(Region \#5)} \\
[-6.4159 \;\; -4.6953]x + 0.6423 & \text{if} \begin{bmatrix} -6.4159 & -4.6953 \\ -0.0275 & 0.1220 \\ 6.4159 & 4.6953 \end{bmatrix} x \leqslant \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix}, \\
& \text{(Region \#6)} \\
-2.0000 & \text{if} \begin{bmatrix} 3.4155 & -4.6452 \\ -0.1044 & -0.1215 \\ -0.1259 & -0.0922 \end{bmatrix} x \leqslant \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix}, \\
& \text{(Region \#7, \#8)} \\
[-6.4159 \;\; -4.6953]x - 0.6423 & \text{if} \begin{bmatrix} 6.4159 & 4.6953 \\ 0.0275 & -0.1220 \\ -6.4159 & -4.6953 \end{bmatrix} x \leqslant \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix} \\
& \text{(Region \#9)}
\end{cases}$$

and consists of $N_{\text{mpc}} = 7$ regions. Region #1 corresponds to the unconstrained linear controller, regions #2, #3, #4 and #5, #7, #8 correspond to the saturated controller, and regions #6 and #9 are transition regions between the unconstrained and the saturated controller. Note that the mp-QP solver provides three different regions #2, #3, #4, although in all of them, $u = u_t^* = 2$. The reason for this is that the second component of the optimal solution, $u_{t+1}^*$, is different, in that $u_{t+1}^* = [-3.4155 \;\; 4.6452]x(t) - 0.6341$ in region #2, $u_{t+1}^* = 2$ in region #3, and $u_{t+1}^* = -2$ in region #4. Moreover, note that regions #2 and #4 are joined, as their union is a convex set, but the same cannot be done with region #3, as their union would not be a convex set, and therefore cannot be expressed as one set of linear inequalities.

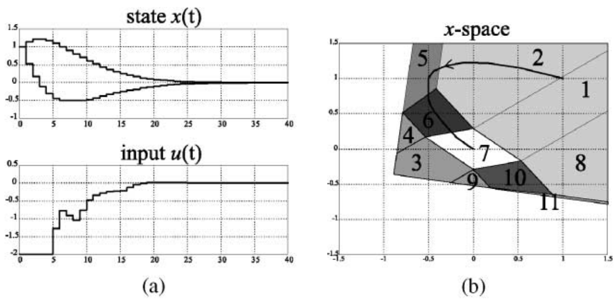


Fig. 3. Example 7.1. Additional constraint $x_{t+k|t} \geqslant -0.5$: (a) closed-loop MPC; (b) state-space partition and closed-loop MPC trajectories.

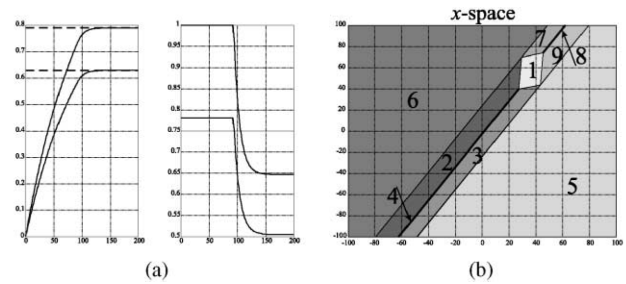


Fig. 4. MIMO example: (a) closed-loop MPC: output $y(t)$ (left), input $u(t)$ (right); (b) state-space partition obtained by setting $u = [0\ 0]'$ and $r = [0.63\ 0.79]'$.

The same example is repeated with the additional state constraint $x_{t+k|t} \geqslant x_{\min}$,

$$x_{\min} \triangleq \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix},$$

$k = 1$. The closed-loop behavior from the initial condition $x(0) = [1\ 1]'$ is depicted in Fig. 3(a). The MPC controller was computed in 0.99 s (22 regions examined). The polyhedral partition of the state space corresponding to the modified MPC controller is depicted in Fig. 3(b). The partition provided by the mp-QP algorithm consists now of $N_r = 11$ regions (as regions #1, #2 and #3, #9 can be joined, the MPC controller consists of $N_{\text{mpc}} = 9$ regions). Note that there are feasible states smaller than $x_{\min}$, and vice versa, infeasible states $x \geqslant x_{\min}$. This is not surprising. For instance, the initial state $x(0) = [-0.6,0]'$ is feasible for the MPC controller (which checks state constraints at time $t + k$, $k = 1$), because there exists a feasible input such that $x(1)$ is within the limits. On the contrary, for $x(0) = [-0.47, -0.47]$ no feasible input is able to produce a feasible $x(1)$. Moreover, the union of the regions depicted in Fig. 3(b) should not be confused with the region of attraction of the MPC closed-loop. For instance, by starting at $x(0) = [46.0829, -7.0175]'$ (for which a feasible solution exists), the MPC controller runs into infeasibility after $t = 9$ time steps.

### 7.2. Reference tracking for a MIMO system

Consider the plant

$$y(t) = \frac{10}{100s + 1} \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} u(t), \tag{31}$$

which was studied by Mulder, Kothare, and Morari (1999) and Zheng, Kothare and Morari (1994) and by other authors as an example for anti-windup control synthesis. The input $u(t)$ is subject to the saturation constraints

$$-1 \leqslant u_i(t) \leqslant 1, \quad i = 1, 2.$$

Mulder et al. (1999) use a decoupler and two identical PI controllers

$$K(s) = \left(1 + \frac{1}{100s}\right) \begin{bmatrix} 2 & 2.5 \\ 1.5 & 2 \end{bmatrix}.$$

For the set-point change $r = [0.63, 0.79]'$ they show that very large oscillations result during the transient when the output of the PI controller saturates.

We sample the dynamics (31) with $T = 2$ s, and design an MPC law (24) with $N_y = 20$, $N_u = 1$, $N_c = 0$, $Q = I$, $R = 0.1I$. The closed-loop behavior starting from zero initial conditions is depicted in Fig. 4(a). It is similar to the result reported by Mulder et al. (1999), where an anti-windup scheme is used on top of the linear controller $K(s)$.

The mp-QP problem associated with the MPC law has the form (7) with two optimization variables (two inputs over the one-step control horizon), and six parameters (two states of the original system, two states to memorize the last input $u(t - T_s)$, and two reference signals), with

$$H = \begin{bmatrix} 0.7578 & -0.9699 \\ -0.9699 & 1.2428 \end{bmatrix},$$

$$F = \begin{bmatrix} 0.1111 & -0.1422 \\ -0.0711 & 0.0911 \\ 0.7577 & -0.9699 \\ -0.9699 & 1.2426 \\ -0.1010 & 0.1262 \\ 0.0757 & -0.1010 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \qquad W = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The solution was computed in 1.15 s (13 regions examined) with Algorithm 1, and the explicit MPC controller is defined over $N_{\text{mpc}} = 9$ regions.

A section of the $x$-space of the piecewise affine solution obtained by setting $u = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and $r = \left[\begin{smallmatrix}0.63\\0.79\end{smallmatrix}\right]$ is depicted in Fig. 4(b). The solution can be interpreted as follows. Region #1 corresponds to the unconstrained optimal controller. Regions #4, #5, #6, #9 correspond to the saturation of both inputs. For instance in region #4 the optimal input variation $\delta u(t) = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] - u(t-1)$. In regions #2, #3, #7, #8 only one component of the input vector saturates. Note that the component which does not saturate depends linearly on the state $x(t)$, past input $u(t-1)$, and the reference $r(t)$, but with different gains than the unconstrained controller of region #1. Thus, the optimal piecewise affine controller is not just the simple saturated version of a linear controller.

In summary, for this example, the mp-QP solver determines: (i) a two-degree of freedom optimal controller for this MIMO system (31); and (ii) a *set-point dependent* optimal anti-windup scheme, a nontrivial task as the regions in the $(x, r, u)$-space where the controller should be switched must be determined. We stress the fact that optimality refers precisely to the design performance requirement (24).

### 7.3. Infinite horizon LQR for the double integrator

Consider the double integrator

$$y(t) = \frac{1}{s^2} u(t)$$

and its equivalent discrete-time state-space representation

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = [1\ 0] x(t), \tag{32}$$

obtained by setting $\ddot{y}(t) \approx (\dot{y}(t+T) - \dot{y}(t))/T$, $\dot{y}(t) \approx (y(t+T) - y(t))/T$, $T = 1$ s.

We want to regulate the system to the origin while minimizing the quadratic performance measure

$$\sum_{t=0}^{\infty} y'(t)y(t) + \frac{1}{10} u^2(t) \tag{33}$$

subject to the input constraint

$$-1 \leqslant u(t) \leqslant 1. \tag{34}$$

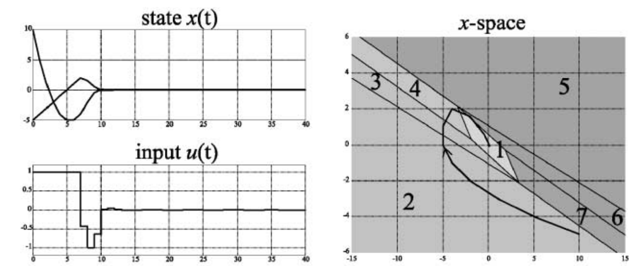


Fig. 5. Double integrator example ($N_u = 2$, $N_{\text{mpc}} = 7$, computation time 0.77 s): (a) closed-loop MPC; (b) polyhedral partition of the state-space and closed-loop MPC trajectories.

This task is addressed by using the MPC algorithm (3) where $N_y = 2$, $N_u = 2$,

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$R = 0.01$, and $K, P$ solve the Riccati equation (5). For a certain set of initial conditions $\chi(0)$, this choice of $P$ corresponds to setting $u_{t+k} = K_{\text{LQ}} x_{t+k|t} = [-0.81662\ \ -1.7499] x_{t+k|t}$ and minimizes

$$\sum_{k=0}^{\infty} y'_{t+k|t} y_{t+k|t} + \frac{1}{10} u^2_{t+k}. \tag{35}$$

On $\chi(0)$, the MPC controller coincides with the constrained linear quadratic regulator, and is therefore stabilizing. Its domain of attraction is, however, larger than $\chi(0)$. We test the closed-loop behavior from the initial condition $x(0) = [10\ -5]'$, which is depicted in Fig. 5(a).

The mp-QP problem associated with the MPC law has form (7) with

$$H = \begin{bmatrix} 1.6730 & 0.7207 \\ 0.7207 & 0.4118 \end{bmatrix}, \qquad F = \begin{bmatrix} 0.9248 & 0.3363 \\ 2.5703 & 1.0570 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \qquad W = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and was computed in 0.77 s (16 regions examined). The corresponding polyhedral partition of the state-space is depicted in Fig. 5(b).

The same example was solved by increasing the number of degrees of freedom $N_u$. The corresponding partitions, computation times, and number of regions are reported in Fig. 6. Note that by increasing the number of free control moves $N_u$, the control law only changes far away from the origin, the more in the periphery the larger $N_u$. This must be expected from the results of Section 3, as the set where the MPC law approximates the

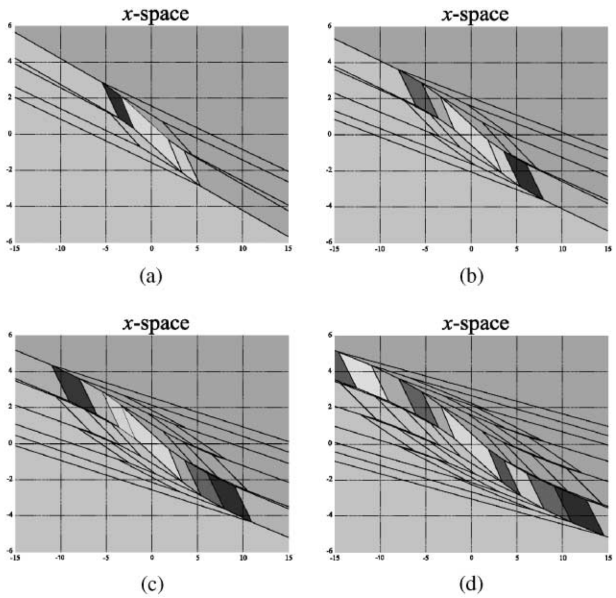


Fig. 6. Partition of the state space for the MPC controller: $N_u$ = number of control degrees of freedom, $N_{\mathrm{mpc}}$ = number of polyhedral regions in the controller, off-line computation time: (a) $N_u = 3$, $N_{\mathrm{mpc}} = 15$, CPU time: 2.63 s; (b) $N_u = 4$, $N_{\mathrm{mpc}} = 25$, CPU time: 560 s; (c) $N_u = 5$, $N_{\mathrm{mpc}} = 39$, CPU time: 9.01 s; (d) $N_u = 6$, $N_{\mathrm{mpc}} = 57$, CPU time: 16.48 s.

constrained infinite-horizon linear quadratic regulation (C-LQR) problem gets larger when $N_u$ increases (Chmielewski & Manousiouthakis, 1996; Scokaert & Rawlings, 1998). By the same arguments, the *exact* piecewise affine solution of the C-LQR problem can be obtained for any set of initial conditions by choosing the finite horizon as outlined in Section 3.

Preliminary ideas about the shape of the constrained linear quadratic regulator for the double integrator were presented in Tan (1991), and are in full agreement with our results. By extrapolating the plots in Fig. 6, where the band of unsaturated control actions is partitioned in $2N_u - 1$ sets, one may conjecture that as $N_u \to \infty$, the number of regions of the state-space partition tends to infinity as well. Note that although the band of unsaturated control may shrink asymptotically as $\|x\| \to \infty$, it cannot disappear. In fact, such a gap is needed to ensure the continuity of the controller proved in Corollary 2. The fact that more degrees of freedom are needed as the state gets larger in order to preserve LQR optimality is also observed by Scokaert and Rawlings (1998), where for the state-space realization of the double integrator chosen by the authors the state $x = [20{,}20]$ requires $N_u = 33$ degrees of freedom.

## 8. Conclusions

We showed that the linear quadratic optimal controller for constrained systems is piece-wise affine and we provided an efficient algorithm to determine its parameters. The controller inherits all the stability and performance properties of model predictive control (MPC) but can be implemented without any involved on-line computations. The new technique is not intended to replace MPC, especially not in some of the larger applications (systems with more than 50 inputs and 150 outputs have been reported from industry). It is expected to enlarge its scope of applicability to situations which cannot be covered satisfactorily with anti-windup schemes or where the on-line computations required for MPC are prohibitive for technical or cost reasons, such as those arising in the automotive and aerospace industries. The decision between on-line and off-line computations must be related also to a tradeoff between CPU (for computing QP) and memory (for storing the explicit solution). Moreover, the explicit form of the MPC controller allows to better understand the control action, and to analyze its performance and stability properties (Bemporad, Torrisi, & Morari, 2000). Current research is devoted to develop on-line implementation techniques which do not require the storage of the polyhedral cells (Borrelli, Baotic, Bemporad, & Morari, 2001), and to develop suboptimal methods that allow one to trade off between performance loss and controller complexity (Bemporad & Filippi, 2001).

All the results in this paper can be extended easily to 1-norm and $\infty$-norm objective functions instead of the 2-norm employed in here (Bemporad, Borrelli, & Morari, 2000). The resulting multiparametric linear program can be solved in a similar manner as suggested by Borrelli et al. (2000) or by Gal (1995). For MPC of hybrid systems, an extension involving multiparametric mixed-integer linear programming is also possible (Bemporad, Borrelli, & Morari, 2000; Dua & Pistikopoulos, 2000). Finally, we note that the semi-global stabilization problem for discrete-time constrained systems with multiple poles on the unit circle which has received much attention since the early paper by Sontag (1984) can be addressed in a completely general manner in the proposed framework.

## Acknowledgements

This research was supported by the Swiss National Science Foundation. We wish to thank Nikolaos A. Bozinis for his help with the initial implementation of the algorithm, and Francesco Borrelli and Ali H. Sayed for suggesting improvements to the original manuscript.

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