The Burnside Di-Lemma: Combinatorics and Puzzle Symmetry

Nick Baxter

Introduction

Combinatorics, the mathematics of counting, provides invaluable tools for both puzzle solving and puzzle design. Solvers of mathematical and mechanical puzzles are often confronted with difficult issues of counting combinations, often complicated by symmetry. Similarly, puzzle designers may want to add elegance to their designs by incorporating symmetry and using sets of pieces that are somehow aesthetically pleasing in their completeness (such as the so-called *English Selection*[1]).

Conventional techniques are not always sufficient to solve some combinatorial problems, especially those where symmetry reduces the number of unique configurations. Fortunately, there is a particularly powerful, but relatively unknown tool for exactly this type of problem: the *Pólya-Burnside Lemma*. This paper will present this principle in common language, and give specific examples of how it can be used.

Nick Baxter is a mathematician and puzzle expert who loves to count.

[1] James Dalgety coined the phrase "English Selection," referring to a logically complete set of puzzle pieces. For example, the 12 planar Pentominoes qualify as an English selection; but for most puzzles, such the Soma Cube, Instant Insanity, Tangrams, Eternity, etc., this is not the case. Many times, whether or not a set is an English selection is a judgment call, and can be artificially contrived since the reference domain can be arbitrary constructed.

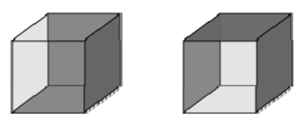
2-Color Cube Problem

Let's say we want to determine the total number of possible 2-color Instant Insanity cubes–in other words, how many different ways can you paint the faces of a cube using no more than two specific colors, say black and white?

At first blush, it may seem as if the answer is simply 2^6, since the cube has six faces, each of which can be colored two ways. But that answer double-counts certain colorings. For example, the coloring in which two opposite faces of the cube are black and the others white gets counted three times: once when the black faces are top and bottom, again when they're front and back, and once more when they're left and right. So 64 is really just an *upper bound* on the answer we're looking for.

The trick is to organize the counting so that every possible configuration is counted *exactly once*. One approach is to first inventory the ways to partition the faces into at most two like-colored sets, and then later apply the colors to the patterns found:

- 6-0: Clearly, there's just one way to have six faces of one color and none of the other (we'll worry about which color it is later).
- 5-1: There is only one way to have just one face colored different from the other five (remember, we are ignoring rotations).
- 4-2: There are two ways to do this. One has the "minority" color on opposites faces of the cube; the other has it on two adjacent faces.
- 3-3: There are two ways to do this also. One has like-colored faces all meeting at a common vertex (with the remaining faces similarly oriented at the opposite vertex); the other has like-colored faces in a row.



Now applying two colors to each of these patterns, we find 2, 2, 4, and 2 colorings respectively (because of symmetry, one must be careful not to count each of the 3-3 patterns twice), giving a total of 10, which is considerably less than our upper bound of 64.

For this problem, we had to be somewhat careful, but symmetry didn't cause too much trouble. The results would be the same if we "colored" the cube's faces with any two distinct markings, as long as both the markings had 90° rotational symmetry, such as ⊠ and ⧇. But what if we used ⊟ and ◫ instead? Now the trouble with symmetry becomes apparent: do we have two patterns or just one? Approaching the problem by inventorying the partitions of faces no longer works; and simply determining if two cubes are really the same becomes much more difficult to visualize. To better understand the difficulty, try solving this new problem; it will be discussed later in the examples.

These two problems can be extended to four (or more) colors or orientations. With four options per face, there are 4^6 (4,096) total permutations. With so many possibilities, these problems move outside most puzzlers' scope of reliably determining inventories and symmetries by hand. One could write a search program, but such techniques are not necessarily of interest to mathematicians or non-programmers.

Pólya-Burnside Lemma

The good news is that the Pólya-Burnside Lemma[1] is the perfect tool for solving this type of problem. It's been known to mathematicians for over 100 years, but surprisingly I've found that many puzzlers, including those well-versed in recreational mathematics, are not familiar with this powerful technique. The rest of this paper presents the Pólya-Burnside Lemma without the jargon of combinatorial group theory, and demonstrates how easy it is to use this powerful tool.

In the world of pure mathematics, any theorem must be precisely stated, including the conditions for when the theorem may be applied. Perhaps one reason why puzzlers don't know the usefulness of the Lemma is that it is not always stated in terms that are easy to understand. For example, consider this version, taken from Eric Weisstein's *MathWorld* [11] (where it's called the Cauchy-Frobenius Lemma):

[1] Within mathematical circles, there has been plenty of discussion regarding the proper name and attribution, and it is probably best known recently as *Burnside's Lemma*. Neumann [8] gives an excellent history of this and a compelling case for the name *Cauchy-Frobenius Lemma.* However, I've chosen to recognize those that first applied the underlying principles to combinatorics, and to use a name that appears to be most familiar with the intended audience.

> Let *J* be a finite group and the image *R(J)* be a representation which is a homeomorphism of *J* into a permutation group *S(X)*, where *S(X)* is the group of all permutations of a set *X*. Define the orbits of *R(J)* as the equivalence classes under $x \sim y$, which is true if there is some permutation *p* in *R(J)* such that $p(x) = y$. Define the fixed points of p as the elements *x* of *X* for which $p(x) = X$. Then the average number of fixed points of permutations in *R(J)* is equal to the number of orbits of *R(J)*.

Outside the world of group theory, this doesn't help much. Better is the concise statement from a text on combinatorial mathematics by C. L. Liu [6]:

> **Theorem (Burnside)**
>
> The number of equivalence classes into which a set *S* is divided by the equivalence relation induced by a permutation group *G* of *S* is given by
>
> $$\frac{1}{|G|}\sum_{\pi \in G}\psi(\pi)$$
>
> where $\psi(\pi)$ is the number of elements that are invariant under the permutation π.

This may be easy for mathematicians to understand, but not for the rest of us. Let's start by translating the terminology it into language that a puzzler can understand.

S **-** The set of all possible variations of an object, with no considerations for rotations and reflections. For painting a cube with four colors, this set has 4^6 members.

Permutation **-** For puzzles, the permutations of interest are those that take a physical object and reorient it so that it appears structurally the same (ignoring coloring or other variations to be considered later). For a cube, a permutation would be any way to pick it up and put it back down in the same place. When doing so, any of the six faces can be face down; then there are four ways to rotate the cube so that the bottom face stays on the bottom. Thus there are 24 possible permutations of a cube.

It is important to remember that a permutation is the *action* that transforms the cube to a new position, not the position itself.

Permutation Group **-** This is the set of all possible permutations of the physical objects under consideration. It forms a *group* in the formal mathematical sense because the result of applying one permutation and then another is again a permutation of the objects. $|G|$ is the short-hand notation for the number of permutations in group G.

Equivalence Relation **-** This is the rule that determines whether or not two objects are the same. An "equivalence relation induced by a permutation group" is simply saying that two objects are the same if there is a permutation that transforms one into the other. So this is really what puzzlers mean when they say that two objects are "the same" or use phrases like "ignoring rotations and reflections."

Equivalence Class - When considering permutations of physical objects, this is a set of objects that are the same, ignoring rotations and reflections. When one is looking for the number of "unique shapes" or "unique colorings", we are really looking for the number of equivalence classes.

Invariant Element **-** This is any object that appears exactly the same before and after a permutation. For example, the first tile below is invariant when rotated 0°, 90°, 180°, and 270°; the second tile is invariant when rotated 0° and 180°; and the third tile is invariant only when not rotated at all.

4-way 2-way 1-way

The most important concept is that of invariant elements because the Pólya-Burnside Lemma reduces all problems of symmetry to simply counting the number of invariant elements for each permutation. The key is that for many puzzles, this counting is significantly easier than any other equivalent problem-solving technique.

So it makes sense to first consider a *base object*, such as a cube, domino, tile, rectangle, etc., without any of the alternations or reorientations prescribed by the puzzle. Next, a *puzzle object* is a member of the set of all variations of the base object that satisfy the puzzle's constraints. Thus the typical puzzle will ask for the number of *unique* puzzle objects satisfying the given criteria.

Now we can restate the Pólya-Burnside Lemma using language that puzzlers can use.

Pólya-Burnside Lemma—Puzzlers' Version

The number of unique puzzle objects that are variations of a base object *p* is

$$\frac{1}{n}(\psi(\pi_1) + \psi(\pi_2) + \ldots + \psi(\pi_n))$$

where $G = \{\pi_1, \pi_2, \ldots, \pi_n\}$ is the set of all physical permutations of *p*, and $\psi(\pi)$ is the number of invariant puzzle objects for the permutation π.

It may seem as if the Polya-Burnside Lemma simply turns one counting problem into a multitude, since the number of permutations can itself be huge. But in theory–and in practice–permutations can be organized into families of similar operations, with the same $\psi(\pi)$ for each member in each family. (In the jargon of group theory, the families are called conjugacy classes.) This often reduces a difficult counting problem to just a handful of easily solved counting problems.

All About the Cube

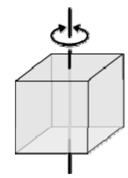
Let's return to the first example, coloring a cube with two colors, and see how this works. The base object *p* is just a normal cube, and *G* is the corresponding set of 24 cube permutations, so n=24. The cube permutations can be grouped into families of similar operations as follows:

$G = \{I, Q, H, D, E\}$, where

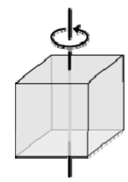
- *I*: Identity–no rotation (1 permutation)
- *Q*: 90° face rotations (6 permutations—three pairs of opposite faces, rotation in either direction)

- *H*: 180° face rotations (3 permutations—three pairs of opposite faces)
- *D*: 120° major diagonal rotations (8 permutations—four pairs of opposite vertices, rotation in either direction)
- *E*: 180° center-edge rotation (6 permutations—six pairs of opposite edges)

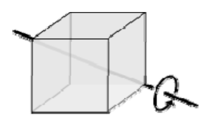
For *I*, there are 2^6 ways to color the cube with two colors (ignoring rotations, reflections, or any other reorientation of the puzzle object). Since every object is invariant under the identity permutation, the total for this case is still 2^6.



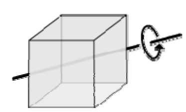
For each of the six *Q* permutations, both the top and bottom faces can be any color, since they stay in the same location during the rotation. For the four side faces, a *Q* permutation rotates one face to the next in a cycle of four. Thus they must all be the same color if a 90° rotation is to appear exactly the same. This gives 2^3 invariant objects.



For each of the three *H* permutations, again the top and bottom faces can be any color. But unlike *Q*, an *H* permutation rotates each side face to the opposite face. Thus each opposite pair can be colored independently, and still leave the cube invariant after rotation. This gives 2^4 invariant objects.



For each of the eight *D* permutations, the cube is rotated around a major diagonal and the two vertices it connects. The three faces touching each of those two vertices must be the same color. This gives 2^2 invariant objects.



For each of the six *E* permutations, the two faces adjacent to the edges that rotate must be the same. The top and bottom face must also be the same color. This gives 2^3 invariant objects.

In total, the Pólya-Burnside Lemma gives

$$\frac{1}{24}\left(\psi(I)+6\psi(Q)+3\psi(H)+8\psi(D)+6\psi(E)\right)$$

$$= \frac{1}{24}\left(2^6 + 6\cdot 2^3 + 3\cdot 2^4 + 8\cdot 2^2 + 6\cdot 2^3\right)$$

or 10 ways, agreeing with the previous solution. For this particular problem, using the Pólya-Burnside Lemma may have been more work than otherwise, but we should feel a lot better about not double-counting or missing any special cases.

Now for the magic—let's consider the same problem but with k colors instead of just 2. Using the Pólya-Burnside Lemma, the analysis is almost identical to that of the 2-color case:

$\psi(I) = k^6$, $\psi(Q) = k^3$, $\psi(H) = k^4$, $\psi(D) = k^2$, and $\psi(E) = k^3$.

Thus the total number of unique colorings is

$$\left(k^6 + 3k^4 + 12k^3 + 8k^2\right)/24$$

This result would have been very difficult[1] or impossible without the use of the Pólya-Burnside Lemma.

Examples

We've shown that the Pólya-Burnside Lemma is general-purpose, relatively fast, and highly reliable. More important, it can help solve problems that would otherwise be next to impossible to solve. To demonstrate this power, I encourage you to imagine solving each of the following example problems using some other technique.

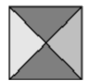
Second Cube Example - How many unique ways are there to paint a cube with ⊟ on each face?

The two orientations of ⊟ behave like two distinct colors, and the analysis of invariant objects is the same as before—except for Q, the family of 90° rotations. In this case, the top and bottom faces must have 90° rotational symmetry if the cube is to be an invariant object. But since ⊟ does not have this symmetry, the count for Q is zero. So instead, the total number of unique cubes is

[1] This is different from the formula originally given by Gardner [3] in the chapter *The Calculus of Finite Differences* (but corrected in subsequent editions). That formula strangely worked only for cases n=1, 2, 3, and 6—perhaps demonstrating the risk of relying solely on empirical results and finite differences for such problems.

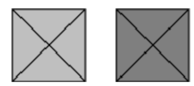
$$\frac{1}{24}\left(2^6 + 6 \cdot 0 + 3 \cdot 2^4 + 8 \cdot 2^2 + 6 \cdot 2^3\right) = 8$$

Edge-Matching Tiles - How many different square edge-matching tiles are there using at most k colors?

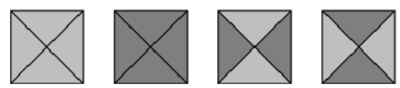


Assuming the tiles are one-sided, there are 4 permutations which can be grouped into three familiar categories: I, Q (2 cases), and H.

For I, there are k^4 invariant objects.



For Q, the four quadrants must all be the same color if the tile is to be invariant after a 90° rotation. Thus there are just k invariant objects. (Illustrated for k=2.)



For H, a 180° rotation swaps pairs of opposite quadrants; thus there are k^2 invariant objects. (Illustrated for k=2. Remember, when counting invariant objects, we ignore rotations; so the third and fourth figures are counted separately.)

The total number of colored tiles is

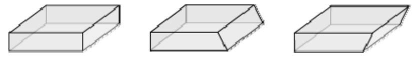
$$\frac{1}{4}\left(\psi(I) + 2\psi(Q) + \psi(H)\right)$$

$$= \frac{1}{4}\left(k^4 + 2k + k^2\right)$$

Beveled Tiles[1] - In 3D, a square tile has a top face, a bottom face, and four side faces. In a beveled tile, each of the four side faces can have one of three styles: flat, angled in, or angled out. How many unique beveled tiles are possible?

---

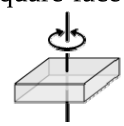
[1] This problem was posed by Ed Pegg, Jr. at the 21st International Puzzle Party–Tokyo, in preparation for a puzzle design he was considering.



A plain square tile is a flattened cube, and has only eight permutations: *I*, *Q* (2 cases), *H* (3 cases), and *E* (2 cases). Because not all the faces are square, there are actually two "flavors" of *H* that must be considered separately: H_1 is a 180° rotation about the center of the square face; H_2 includes the two 180° rotations about the center of a side face.

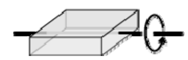
For *I*, each side face can have one of three styles, for 3^4 objects.

For *Q*, the tile can be rotated 90° about the center of the square face in either direction. To be invariant, a configuration must have the same style on all four side faces. Thus there are just 3 invariant objects.



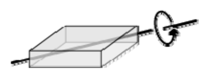
Q and H_1

For H_1, each pair of opposite side faces can have any of the three styles, giving 3^2 invariant objects.



H_2

For H_2, the two side faces that stay in the same place must be the flat style in order to appear the same when turned upside-down. The remaining two side faces must be paired, giving 3 invariant objects.



E

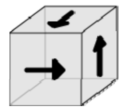
For *E*, the tile is rotated 180° about an axis connecting the centers of opposite short edges. One of the pair of side faces adjacent to such an edge can be any of the three styles, and will determine the style of the other. The same is true for the opposite edge, giving 3^2 invariant objects.

Thus, the total number of unique tiles is

$$\frac{1}{8}\left(\psi(I)+2\psi(Q)+\psi(H_1)+2\psi(H_2)+2\psi(E)\right)$$

$$=\frac{1}{8}\left(3^4+2\cdot 3+3^2+2\cdot 3+2\cdot 3^2\right)=15$$

Four Arrows[1] - How many different ways can you put six arrows on the faces of a cube? (The arrows must be in one of four orthogonal orientations).



The analysis of invariant objects is the same as the *k*-color cube problem (for k=4), except that we must be more careful with the orientation of the arrows.

Case I is the same as before: 4^6.

For Q, and H, the top and bottom faces must be invariant after 90° or 180° rotations, respectively. Since the arrow does not have such symmetry, there are no invariant objects for these permutations.

For D, there are two cycles of three faces. Within each cycle, the arrow on one face can be any orientation, forcing the orientation for each of the other two (see faces shown in the above figure). Thus there are 4^2 invariant objects.

For E, there are three pairs of faces that cycle. Similar to D, each cycle can have four arrow orientations, giving 4^3 invariant objects.

The total number of cubes is

$$\frac{1}{24}\left(\psi(I)+6\psi(Q)+3\psi(H)+8\psi(D)+6\psi(E)\right)$$

$$=\frac{1}{24}\left(4096+6\cdot 0+3\cdot 0+8\cdot 16+6\cdot 64\right)=192$$

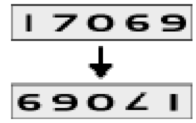
In all our examples so far, the permutations have all corresponded in a clear-cut way to some sort of physical movement. We finish with one more example in which the notion of permutation is somewhat more subtle.

Numbered Slips - How many slips of paper are needed to individually print all (zero-filled) *n*-digit numbers?

---

[1] This problem was most recently posed by Moscovich [7], problem #200.

This problem is tricky because we cannot use the expected permutation group of {*I*, *H*}—a 180° rotation turns some numbers into invalid symbols, thus making it invalid as a permutation (see example).



Instead we must come up with an alternative permutation. C. L. Liu [6] defines *H'* as a 180° rotation for those numbers containing only the digits 0,1,6,8,9; otherwise it is the same as *I*. This is an improvement over *H*, since *H'* always gives a valid result. And since applying *H'* twice gives *I*, one can easily show that {*I*, *H'*} is in fact a mathematically proper permutation group. With this, we can get back to looking at invariant objects.

For *I*, the total of invariant objects is 10^n.

For *H'*, by definition, all numbers not made of only the symmetry digits (0, 1, 6, 8, 9) are clearly invariant. This is 10^n-5^n slips. For numbers using only symmetric digits, we must consider even and odd values of *n* separately.

For even *n*, each of the first *n*/2 digits can be any of five symmetric digits, forcing the selection of the last n/2 digits. This gives $5^{n/2}$ invariant objects.

For odd *n*, each of the first (*n*-1)/2 digits can be any of the five symmetric digits, forcing the selection of the last (*n*-1)/2 digits. The middle digit can be one of just three digits (0,1,8) with 180° symmetry, giving a grand total of $3 \cdot 5^{(n-1)/2}$ invariant objects.

The total number of slips is $\frac{1}{2}\left(\psi(I) + \psi(H')\right)$

$$= \begin{cases} 10^n - \dfrac{\left(5^n - 3 \cdot 5^{(n-1)/2}\right)}{2}, \ n \ odd \\ 10^n - \dfrac{\left(5^n - 5^{n/2}\right)}{2}, \ n \ even \end{cases}$$

Beyond Pólya-Burnside

The Pólya-Burnside Lemma is actually just a special case of Pólya's Enumeration Theorem [9] (later generalized by de Bruijn [1]). If you know the cycle index of a permutation group, you can create a generating function that gives a pattern inventory, not just the total count.

For example, the cycle index for the permutation group of the six cube faces is

$$\frac{1}{24}\left(x_1^6 + 6x_1^2 x_4 + 3x_1^2 x_2^2 + 8x_3^2 + 6x_2^3\right)$$

For coloring the cube with two colors, represented by *r* and *b*, we simply substitute ($r^i + b^i$) for x_i , giving

$$r^6 + r^5 b + 2r^4 b^2 + 2r^3 b^3 + 2r^2 b^4 + rb^5 + b^6$$

which is the inventory of all possible two-color combinations. Looking back at the 2-Color Cube problem, we see that this corresponds exactly to the inventory that we found by hand, but derived without the risk of missing a case or double-counting.

For further reading, papers of interest not otherwise cited are Burnside [2], Golomb [4], Klass [5], and Read [10].

Acknowledgments

I'd like to acknowledge George Pólya for his inspirational lectures in my undergraduate combinatorics class—at 90 years old. I'd also like to thank Don Knuth and Stan Isaacs for their extensive and constructive comments of the draft, and to David Singmaster, whose inexhaustible enthusiasm rekindled my interest in this area.

References

[1] N. G. de Bruijn, *Generalization of Pólya's Fundamental Theorem in Enumerative Combinatorial Analysis*, Proc. Koninkl. Nederl. Akad. Wetenschap. A **62**, (Indag. Math., **21**), pp. 59-79, 1956.

[2] W. Burnside, *Theory of Groups of Finite Order*, Cambridge University Press, Cambridge, 1911.

[3] Martin Gardner, *Martin Gardner's New Mathematical Diversions from Scientific American*, Simon and Schuster, 1966 (First Printing), pp. 234-246.

[4] Solomon W. Golomb, *Polyominoes*, Princeton University Press, 1994 (revised edition), pp. 43-69.

[5] Michael J. Klass, *A Generalization of Burnside's Combinatorial Lemma*, Journal of Combinatorial Theory (A), **20**, pp. 273-278, 1976.

[6] C. L. Liu, *Introduction to Combinatorial Mathematics*, McGraw-Hill, 1968.

[7] Ivan Moscovich, *1000 PlayThinks*, Workman Publications, 2001.

[8] Peter M. Neumann, *A Lemma that is not Burnside's*, Math. Scientist, **4**, pp. 133-141, 1979.

[9] G. Pólya, *Kombinatorishe Anzahlbestimmungen für Gruppen, Gräphen und chemische Verbindungen*, Acta Math. **68**, pp. 145-254, 1937.

[10] R. C. Read, *Pólya's Theorem and Its Progeny*, Mathematics Magazine, 60(5), pp. 275-282, Dec. 1987.

[11] Eric Weisstein, *World of Mathematics*, http://mathworld.wolfram.com/.